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Free vibration of three-layer circular cylindrical shells with functionally graded middle layer

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ABSTRACT

We study free vibrations of a simply supported three-layer circular cylindrical shell with the inner and the outer layers made of the same homogeneous material and the middle layer composed of a functionally graded material. We use Flügge's shell theory to derive governing equations, express mid-plane displacements in terms of trigonometric functions that identically satisfy the boundary conditions, and compute natural frequencies in terms of the geometrical and the material parameters. Computed results show that the fundamental natural frequency decreases with an increase in the radius-to-thickness ratio, and increases with an increase in the ratio of Young's modulus at the mid-surface to that of the outer (or the inner) layer.

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1. Introduction

Thin circular cylindrical shells are widely used in civil, naval, nuclear, mechanical, chemical and aerospace applications; accordingly their vibrations are of interest to structural engineers. Among numerous works on vibrations of shells we cite a few. Loy et al. (1997) have computed natural frequencies of circular homogeneous cylindrical shells by a generalized differential quadrature method (DQM). Li (2006) studied free vibration of isotropic and orthotropic circular cylindrical shells and delineated the effect of axial pressure on the fundamental frequency. Pellicano (2007) employed both analytical and experimental methods to study linear and nonlinear vibration response of an isotropic cylindrical shell. A local adaptive DQM was used by Zhang et al. (2006) to analyze free vibrations of shells under different boundary conditions. Zhang and Xiang (2006) and Xiang et al. (2002) have studied free vibrations of cylindrical shells with intermediate ring elastic supports.

Vibrations of laminated cross-ply circular cylindrical shells have been studied by Lam and Loy (1995), Zhang (2001), Ganapathi et al. (2003), Jafari et al. (2005) and Wang and Lin (2006). A functionally graded material (FGM) is usually composed of two or more materials in which material properties vary continuously, and a structure made of a FGM is called a FG structure. Haddadpour et al. (2007) have used the Garlerkin method to study free vibrations of a FG cylindrical shell subjected to thermal loadings and obtained frequencies as a function of the temperature rise. Loy et al. (1999) and Pradhan et al. (2000) have investigated the effect of material property, geometry and boundary conditions on free vibration of FG circular cylindrical shells. Recently, Cao and Wang (2007) analytically studied free vibration of a FG cylindrical shell with small holes. Anigeri et al. (2006) used a semi-analytical finite element method to study free vibration of a magneto-electro-elastic cylindrical shell.

Here we study free vibration of a three-layer circular cylindrical shell with the inner and the outer layers made of a all the same and isotropic linear elastic material and the middle layer of an FG linear elastic material with material properties varying continuously in the thickness direction. The motivation for this work is provided by our desire to study free vibrations of a double wall carbon nanotube (DWCNT) by adjusting material properties of the FGM to simulate van der Walls forces among atoms on the two walls of the CNT. We note that Sears and Batra (2006) have studied buckling of DWCNTs. Following Li and Batra (2006) we use Flügge's (1973) shell theory to ascertain the effect on natural frequencies of different material and geometric parameters.

2. Problem formulation

We consider a simply supported three-layer circular cylindrical shell of length l, wall thickness h, and mid-surface radius R, assume that materials of the inner and the outer layers are isotropic, homogeneous and linear elastic, and the middle layer is made of a linear elastic FG material with material properties varying continuously in the thickness direction. The material properties are assumed to

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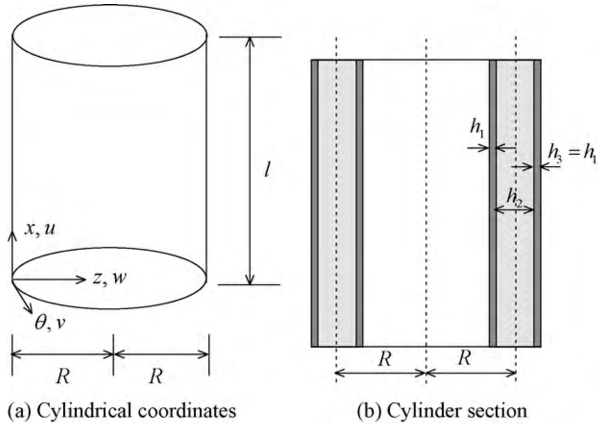

Fig. 1. Details of the geometry of the three-layer circular cylindrical shell.

be continuous at the interfaces between the middle layer and the inner and the outer layers. Thicknesses of the inner, the middle, and the outer layers are h_1, h_2 and h_3, respectively. We describe deformations of the cylinder in cylindrical coordinates x, θ and z in the axial, the circumferential and the thickness directions, respectively, with z and w positive when they point into the cylinder (e.g., see Fig. 1a).

We assume that Poisson's ratio of the FGM is a constant and its Young's modulus varies either as a polynomial of degree one or two in z; that is,

$$E_2(z) = E_1\left(k + 2(1-k)\frac{|z|}{h_2}\right), \quad \left(\left(\frac{-h_2}{2}\right) < z < \left(\frac{h_2}{2}\right)\right) \tag{1}$$

$$E_2(z) = E_1\left(k + 4(1-k)\left(\frac{z}{h_2}\right)^2\right), \quad \left(\frac{-h_2}{2}\right) < z < \left(\frac{h_2}{2}\right) \tag{2}$$

where $k = E_0/E_1$ and $E_0 = E_2(0)$.

Nie and Batra (2010) have shown that the effect of considering spatially varying Poisson's ratio and assuming it to be a constant for a cylindrical pressure vessel loaded by uniform pressure on the inner surface is negligible on stresses and the two displacement fields differ by at most 16%. Several investigators have assumed Poisson's ratio to be constant while studying vibrations of FGMs. This assumption is justified since Poisson's ratios of different materials usually do not differ much. Through-the-thickness variations of Young's modulus given by Eq. (2) are exhibited in Fig. 2a–d.

We refer the reader to Li and Batra (2006) for details of equations governing deformations of the cylinder based on Flügge's (1973) shell theory. Li and Batra (2006) analyzed buckling of a simply supported three-layer cylindrical shell with forces p_x, p_θ and p_z applied per unit area of the middle surface. Governing equations for the dynamic problem can be obtained by setting

$$(p_x, p_\theta, p_z) = \left(-\bar{\rho}\frac{\partial^2 u}{\partial t^2}, -\bar{\rho}\frac{\partial^2 v}{\partial t^2}, -\bar{\rho}\frac{\partial^2 w}{\partial t^2}\right) = (-\bar{\rho}\ddot{u}, -\bar{\rho}\ddot{v}, -\bar{\rho}\ddot{w})$$

where $\bar{\rho} = \int_{-h/2}^{h/2}\rho(z)dz$, ρ equals the mass density (mass/volume) of the shell material, u, v and w are displacements in the x, θ and z-directions, and a superimposed dot indicates differentiation with respect to time t.

3. Harmonic vibrations

For simply supported cylinder edges boundary conditions are identically satisfied by the following displacement field:

$$u = X_1\cos(m\theta)\cos(\lambda_n x/R)\cos(\omega t) \tag{3a}$$

$$v = X_2\sin(m\theta)\sin\left(\frac{\lambda_n x}{R}\right)\cos(\omega t) \tag{3b}$$

$$w = X_3\cos(m\theta)\sin\left(\frac{\lambda_n x}{R}\right)\cos(\omega t) \tag{3c}$$

where $\lambda_n = n\pi R/l$, integers n and m are wave numbers of mode shapes in the axial and the circumferential directions, respectively; X_i ($i = 1, 2, 3$) are unknown constants representing amplitudes of vibration of a mode shape, and ω is a natural frequency of the shell. By introducing Eqs. (3) into governing equations (Li and Batra, 2006), the trigonometric functions drop out and we get the following algebraic equations

$$\left(a_{11} - \frac{\Omega^2}{\lambda_n^2}\right)X_1 + a_{12}X_2 + a_{13}X_3 = 0 \tag{4a}$$

$$a_{21}X_1 + \left(a_{22} - \frac{\Omega^2}{\lambda_n^2}\right)X_2 + a_{23}X_3 = 0 \tag{4b}$$

$$a_{31}X_1 + a_{32}X_2 + \left(a_{33} - \frac{\Omega^2}{\lambda_n^2}\right)X_3 = 0 \tag{4c}$$

for the determination of the non-dimensional natural frequency $\Omega = (\omega R/\lambda_n)\sqrt{\bar{\rho}/C}$ where $C = hE_1/(1-\nu_1{}^2)$ and dimensionless

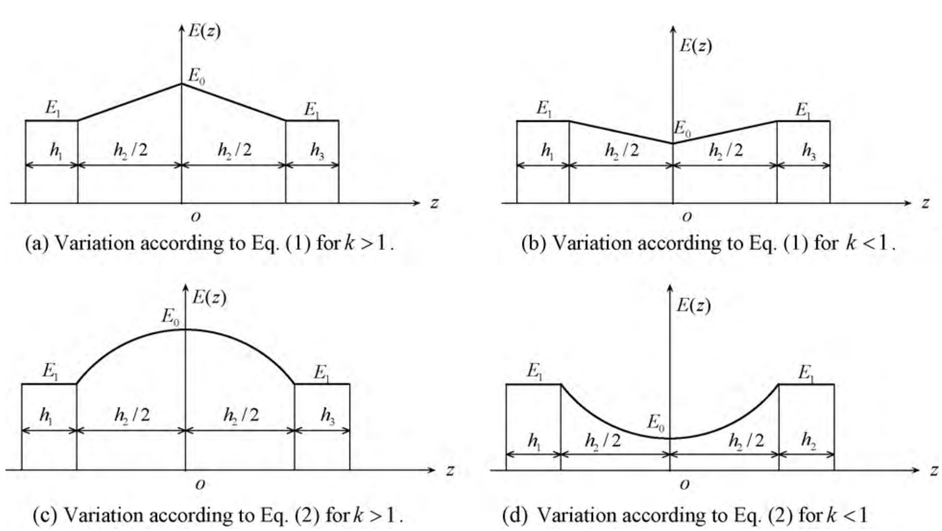

Fig. 2. Through-the-thickness variation of the elastic modulus E of the cylinder materials.

coefficients $a_{ij}(i, j = 1, 2, 3)$ are defined in Li and Batra (2006). The requirement that Eqs. (4a)–(4c) have a non-trivial solution for the amplitude X_i gives an algebraic equation for the determination of Ω.

4. Numerical results

While computing numerical results we consider the case of $h_1 = h_3$, i.e., the inner and the outer cylinders have the same thickness. We use the Newton iteration method to find a root of the algebraic equation for the determination of Ω; this root depends not only on the geometry and the material properties of the shell but also on the wave numbers m and n. In order to verify our algorithm we first consider the case of a cylinder made of a homogeneous material with $k = 1$, $\nu_1 = \nu_2 = 0.3$. It should be clear from the results listed in Table 1 for different values of the axial wave number n that the presently computed non-dimensional natural frequency $\Omega = R\omega\sqrt{(1-\nu^2)\rho/E}$ agrees well with that obtained by Loy et al. (1997, 1999).

We now consider a FG shell made of only one layer (i.e., $\beta = h_2/h = 1$) with Young's modulus given by Eq. (1). For $n = 1$, $m = 4$, values of the dimensionless frequency for different values of the radius/thickness ratio, $\delta = R/h$, and the rigidity ratio, $k = E_0/E_1$, are listed in Table 2. It is observed that the frequency increases with an increase in the value of the parameter k, or equivalently with an increase in the overall stiffness of the middle layer, and decreases with an increase in the values of δ. The incremental decrease in the frequency is noticeable when δ is increased from 100 to 150

Table 1
Comparisons of the non-dimensional frequencies Ω for an isotropic cylindrical shell ($n = 1$, $l/R = 20$, $\delta = 100$, $\nu = 0.3$, $k = 1$).

m	Reference (Loy et al., 1997)	Reference (Loy et al., 1999)	Present
1	0.016101	0.016102	0.016100
2	0.009382	0.009387	0.009383
3	0.022105	0.022108	0.022120
4	0.042095	0.042096	0.042123
5	0.068008	0.068008	0.068054
6	0.099730	0.099730	0.099798
7	0.137239	0.137239	0.137333
8	0.180527	0.180527	0.180652
9	0.229594	0.229594	0.229752
10	0.284435	0.284435	0.284632

Table 2
Dependence of the non-dimensional frequencies $\Omega \times 10^2$ upon parameters δ and k for a FG shell when $n = 1$, $m = 4$, $\beta = 1.0$, $l/R = 5.0$ and material properties given by Eq. (1) with $\nu_1 = \nu_2 = 0.3$.

δ	k					
	0.1	0.2	0.4	0.6	0.8	1.0
100	7.30577	7.58806	8.12321	8.62518	9.09948	9.55021
150	4.20599	4.41010	4.79193	5.14546	5.48763	5.79894
200	3.34081	3.52443	3.87344	4.21586	4.51842	4.78937
250	2.95040	3.15372	3.48886	3.82788	4.10859	4.39992
300	2.77536	2.96778	3.30966	3.62774	3.90422	4.17774
350	2.66555	2.86424	3.20606	3.51426	3.81422	4.07703
400	2.60209	2.78235	3.12170	3.43708	3.70848	3.97761
450	2.55051	2.74562	3.09846	3.39676	3.68808	3.94184
500	2.53686	2.72070	3.07630	3.37596	3.66862	3.93934

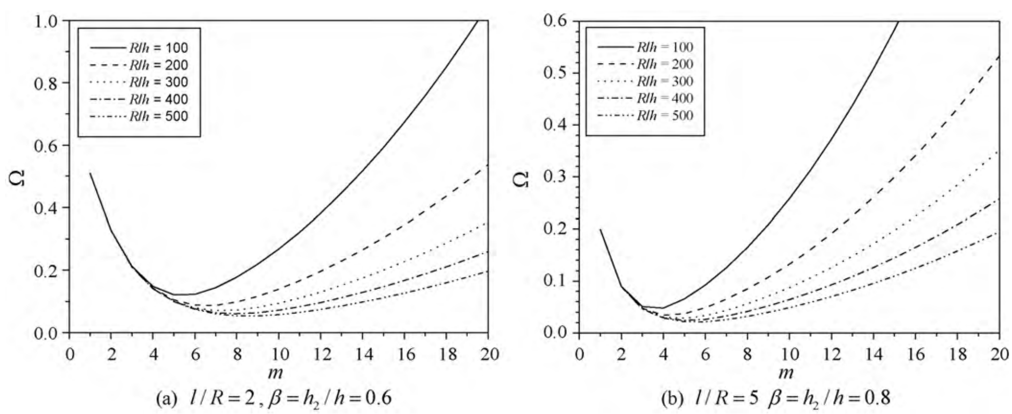

Fig. 3. Non-dimensional frequencies, Ω, versus the wave number, m, for $\delta = R/h$ ($n = 1$, $k = 0.1$) for a FG shell with material properties given by Eq. (1).

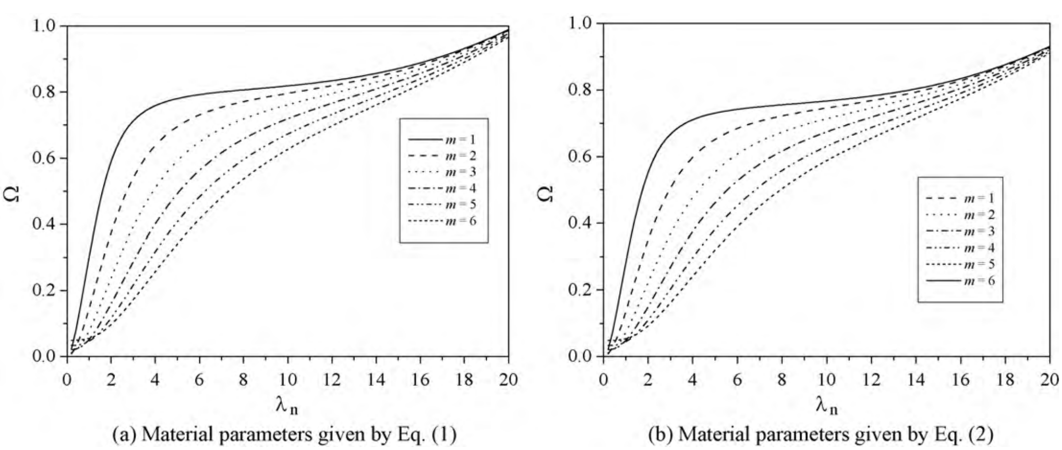

Fig. 4. For circumferential wave number $m = 1, 2, \ldots, 6$, dependence of the non-dimensional frequency, Ω, upon the parameter, λ_n, for the FG cylindrical shell with $\beta = 1$, $\delta = 200$, $k = 0.1$.

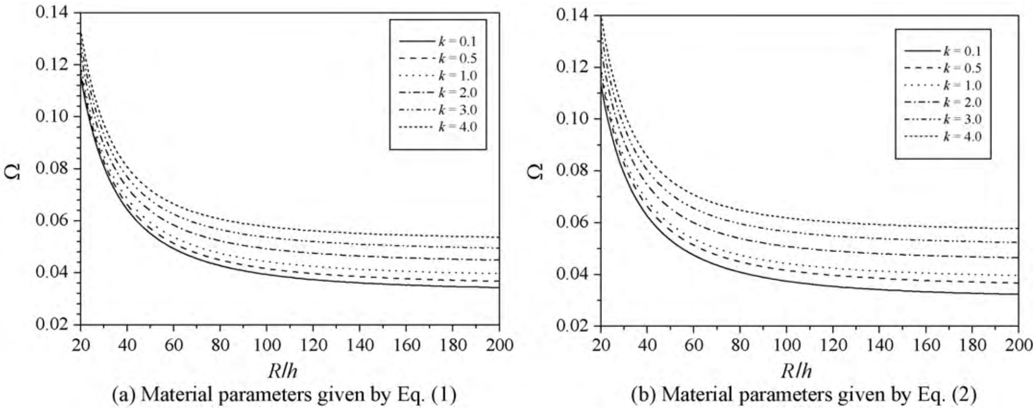

Fig. 5. Natural frequency, Ω, versus radius-to-thickness ratio δ for the tree-layer cylindrical shell for different values of k when $\beta = 0.6$, $l/R = 5$, $n = 1$, $m = 1$.

but subsequent changes in the frequency for every increase in δ by 50 drop rather rapidly signifying a saturation of the frequency as δ becomes very large.

For the three-layer shell with either $h_2 = 0.6h$ or $h_2 = 0.8h$ (i.e., either $\beta = 0.6$ or 0.8) and for the axial wave number $n = 1$, we have plotted in Fig. 3 the variation of the natural frequency, Ω, with the circumferential wave number m for different values of the radius/thickness ratio, R/h. It is evident that the value of m corresponding to the minimum value of Ω increases with an increase in the value of R/h. Furthermore, the minimum value of Ω decreases with an increase in the value of R/h. For a fixed value of m the frequency decreases with an increase in the value of R/h.

We note that for inextensional modes of vibration of free-free zigzag carbon nanotubes studied by Gupta et al. (2009) via molecular mechanics simulations, the frequency saturated with an increase in the value of the circumferential mode number.

For $\beta = 1$, $\delta = 200$, $k = 0.1$, and different values of m, the variation of the frequency with the parameter λ_n is exhibited in Fig. 4. It can be seen that the effect of the wave number m on the frequency is more significant for $\lambda_n < 12$.

In Fig. 5, curves of the dimensionless frequency Ω versus δ are presented for different values of $k = E_0/E_1$ and for the two spatial variations of Young's modulus of the middle layer to illuminate the influence of the parameters δ and k on Ω. These results show that for a fixed value of δ, the frequency increases with an increase in the value of k. Recall that for $k < 1$, the stiffness of the middle layer is less than that of the inner and the outer layers, and the bending rigidity of the middle layer increases with an increase in the value of k. For a fixed k, the frequency is greater for the parabolic variation of E than that for an affine variation of E.

5. Conclusions

Vibration of a simply supported three-layer circular cylindrical shell with functionally graded (FG) middle layer has been studied. Materials of all three layers are isotropic, those of the inner and the outer layers are homogeneous and that of the middle layer is FG with Young's modulus varying continuously in the thickness direction. Governing equations based on Flügge's shell theory are reduced to algebraic equations by assuming expressions for the three displacements that identically satisfy boundary conditions. Effects of the geometric and the material parameters and of the wave number of vibration modes on the frequency have been examined in detail. Numerical results show that the frequency decreases with an increase in the radius/thickness ratio, and increase with an increase in the ratio of Young's modulus at the shell mid-surface to that of the inner or the outer layer.

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