

Material tailoring and universal relations for axisymmetric deformations of functionally graded rubberlike cylinders and spheres

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Abstract

We find the spatial variation of material parameters for pressurized cylinders and spheres composed of either an incompressible Hookean, neo-Hookean, or Mooney–Rivlin material so that during their axisymmetric deformations either the in-plane shear stress or the hoop stress has a desired spatial variation. It is shown that for a cylinder and a sphere made of an incompressible Hookean material, the shear modulus must be a linear function of the radius r for the hoop stress to be uniform through the thickness. For the in-plane shear stress to be constant through the cylinder (sphere) thickness, the shear modulus must be proportional to r^2 (r^3). For finite deformations of cylinders and spheres composed of either neo-Hookean or Mooney–Rivlin materials, the through-the-thickness variation of the material parameters is also determined, for either the in-plane shear stress or the hoop stress, to have a prespecified variation. We note that a constant hoop stress eliminates stress concentration near the innermost surface of a thick cylinder and a thick sphere. A universal relation holds for a general class of materials irrespective of values of material parameters. Here, for axisymmetric deformations, we have derived expressions for the average hoop stress and the average in-plane shear stress, in terms of external tractions and the inner and the outer radii of a cylinder and a sphere, that hold for their elastic and inelastic deformations and for all (compressible and incompressible) materials.

Keywords

functionally graded cylinders and spheres, material tailoring, universal relations

1. Introduction

A challenging problem for structural engineers is to optimize structural geometry and material parameters within the prescribed constraints so that all of the material is fully utilized. One way to ensure this is to have either each stress component or a suitable combination of stresses essentially uniform throughout the body. For a brittle material, failure is usually taken to initiate when the maximum tensile principal stress reaches a critical value. However, for a ductile material, either the von Mises stress or the maximum shear stress governs the failure of the material. Here we study axisymmetric deformations of hollow cylinders and spheres composed of

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rubberlike materials that are assumed to be incompressible. These materials can undergo large elastic deformations, and their failure criteria are not well established. For a given cylinder (sphere) and pressures applied on its inner and outer surfaces, we find the spatial variation of the material parameter that is required to uniformize either the hoop stress or the in-plane shear stress through the cylinder (sphere) thickness. This will inevitably eliminate stress concentration around the innermost surface in a thick cylinder (sphere).

For plane strain axisymmetric deformations of a cylinder composed of an orthotropic compressible material, Leissa and Vagins [1] assumed that all material moduli are proportional to each other and found their spatial variation to make either the hoop stress or the maximum in-plane shear stress uniform through the cylinder thickness. Batra [2] determined the radial variation of the shear modulus in Hookean cylinders and spheres so that the hoop stress was uniform during their axisymmetric deformations. Nie and Batra [3] have computed the spatial variation of the shear modulus for the in-plane shear stress to be uniform through the cylinder thickness with the cylinder made of a Hookean material. However, the approach followed here differs from that employed by Nie and Batra [3] and we give results for finite deformations and derive universal relations. Nie and Batra [4] have also found the radial variation of either Young's modulus or Poisson's ratio for a pressurized cylinder to have either the uniform in-plane shear stress or the uniform hoop stress. Batra [5] studied the torsion of a cylinder made of an incompressible Hookean material with the shear modulus varying along the axial direction, and found the axial variation of the shear modulus to control the angle of twist of a cross-section. Here we study the radial expansion and contraction of an inhomogeneous hollow sphere and a hollow cylinder made of three incompressible materials, namely, Hookean, neo-Hookean, and Mooney–Rivlin. We find the radial variation of the material parameter so as to have a prescribed distribution in the radial direction of either the in-plane shear stress or the hoop stress. Thus by suitably adjusting the spatial variation of the material parameter one can, in principle, minimize the stress concentration near the bounding surfaces of a hollow cylinder and a sphere.

We note that the constitutive relation for an incompressible material involves a hydrostatic pressure that cannot be determined from the deformation field, but is found from the equilibrium equations and boundary conditions. The volume-preserving condition facilitates solving the problem analytically, but poses challenges during the numerical solution of a problem.

Many biological tissues are modeled as incompressible, and the present work will apply to radial expansion/contraction of an artery and a bladder provided that the material can be taken to be isotropic, transient effects are ignored, and they can be approximated as a cylinder and a sphere, respectively.

Problems for inhomogeneous cylinders and spheres have been studied by Horgan and Chan [6], Fang and Long [7], Jabbari et al. [8], Shao and Ma [9], Lechnitskii [10], Timoshenko and Goodier [11], Pan and Roy [12], Liew et al. [13], Tarn [14], Tarn and Chang [15], Oral and Anlas [16], Obata and Noda [17], Kim and Noda [18], Chatzigeorgiou et al. [19], and Batra [20], amongst others. However, the inverse problem of finding the spatial variation of material parameters for attaining a desired spatial distribution of stresses has received less attention, and this is particularly true for finite deformations.

2. Problem formulation

In the absence of body forces equations, in cylindrical coordinates (r, θ, z) , governing static plane strain axisymmetric deformations of a hollow cylinder subjected to pressures p_{in} and p_{ou} on the inner and the outer surfaces, respectively, are

$$\sigma'_{rr} + (\sigma_{rr} - \sigma_{\theta\theta})/r = 0, \quad (1)$$

$$\sigma_{rr}(r = r_{\text{in}}) = -p_{\text{in}}, \sigma_{rr}(r = r_{\text{ou}}) = -p_{\text{ou}}. \quad (2a, b)$$

Here σ_{rr} is the Cauchy radial stress, $\sigma_{\theta\theta}$ is the Cauchy hoop stress, r_{in} is the inner radius and r_{ou} is the outer radius of the cylinder, and a prime denotes derivative with respect to the radius r .

2.1. Infinitesimal deformations

For a cylinder made of an incompressible and inhomogeneous linear elastic material, σ_{rr} and $\sigma_{\theta\theta}$ are related to the radial strain e_{rr} and the circumferential strain $e_{\theta\theta}$ by

$$\begin{aligned}\sigma_{rr} &= -p + 2\mu(r)e_{rr}, \\ \sigma_{\theta\theta} &= -p + 2\mu(r)e_{\theta\theta},\end{aligned}\quad (3a,b)$$

where p is the hydrostatic pressure not determined by the deformation field, and μ is the shear modulus that depends upon the radial coordinate of a point. For plane strain axisymmetric deformations a cylinder particle moves only radially, and

$$e_{rr} = u', \quad e_{\theta\theta} = u/r, \quad (4)$$

where u is the radial displacement of a point. For deformations to be volume preserving

$$e_{rr} + e_{\theta\theta} = 0. \quad (5)$$

In spherical coordinates (r, θ, ϕ) with the origin at the sphere center, equilibrium equations for axisymmetric deformations of a hollow sphere with pressures p_{in} and p_{ou} , respectively, on the inner and the outer surfaces are

$$\sigma'_{rr} + 2(\sigma_{rr} - \sigma_{\theta\theta})/r = 0, \quad \sigma_{\theta\theta} = \sigma_{\phi\phi}, \quad (6)$$

$$\sigma_{rr}(r = r_{in}) = -p_{in}, \quad \sigma_{rr}(r = r_{ou}) = -p_{ou}. \quad (7a, b)$$

The strain–displacement relations are the same as those given in Equation (4), $e_{\phi\phi} = e_{\theta\theta}$, and for deformations to be isochoric

$$u' + 2u/r = 0. \quad (8)$$

2.2. Finite deformations

For a cylinder or a sphere made of an incompressible neo-Hookean material undergoing finite deformations

$$\sigma_{rr} = -p + 2C_1(r)B_{rr}, \quad \sigma_{\theta\theta} = -p + 2C_1(r)B_{\theta\theta}, \quad (9)$$

$$B_{rr} = \left(\frac{dr}{dR}\right)^2, \quad B_{\theta\theta} = \left(\frac{r}{R}\right)^2. \quad (10)$$

Here r and R are, respectively, the radial coordinates of a point in the current and the reference configurations.

For a cylinder or a sphere composed of a Mooney–Rivlin material

$$\begin{aligned}\sigma_{rr} &= -p + 2C_1(r)B_{rr} + 2C_2(r)/B_{rr}, \\ \sigma_{\theta\theta} &= -p + 2C_1(r)B_{\theta\theta} + 2C_2(r)/B_{\theta\theta},\end{aligned}\quad (11)$$

where $C_1(r)$ and $C_2(r)$ are position-dependent material parameters, and B_{rr} and $B_{\theta\theta} = B_{\phi\phi}$ are given by Equation (10).

3. Universal relations

3.1. Cylinder

Rewriting Equation (1) as

$$(r\sigma_{rr})' - \sigma_{\theta\theta} = 0, \quad (12)$$

integrating it with respect to r over the cylinder thickness, and using boundary conditions (2a, b), we obtain

$$(-r_{ou}p_{ou} + r_{in}p_{in}) = \sigma_{\theta\theta}^a(r_{ou} - r_{in}). \quad (13)$$

Thus the average hoop stress, $\sigma_{\theta\theta}^a$, over the cylinder thickness is determined by pressures applied to the inner and the outer surfaces of the cylinder and their radii. Since Equation (13) holds irrespective of the value of μ , and for all deformations, it is a universal relation valid for plane strain axisymmetric deformations of a hollow cylinder. For a solid cylinder, $r_{\text{in}} = 0$, and Equation (13) gives

$$\sigma_{\theta\theta}^a = -p_{\text{ou}}. \quad (14)$$

Thus the average hoop stress in a solid cylinder with no tractions applied on the outer surface must vanish. Similarly, the average hoop stress is null in a hollow cylinder with either no tractions applied on the inner and the outer surfaces, or when $p_{\text{ou}}r_{\text{ou}} = p_{\text{in}}r_{\text{in}}$. In particular, *the hoop stress in an unloaded hollow cylinder is either zero everywhere or is non-uniform*. These results are valid for cylinders composed of inhomogeneous materials and for all types of axisymmetric deformations.

We now assume that the desired through-the-thickness in-plane shear stress variation is given by

$$\sigma_{\theta\theta} - \sigma_{rr} = g(r), \quad (15)$$

where $g(r)$ is a prescribed smooth function of r . Substitution from Equation (15) into Equation (1), the integration of the resulting equation with respect to r , and the use of boundary conditions (2a, b) gives

$$-p_{\text{ou}} + p_{\text{in}} = \int_{r_{\text{in}}}^{r_{\text{ou}}} \frac{g(r)}{r} dr. \quad (16)$$

Thus the function g must satisfy constraint Equation (16), which involves pressures applied on the inner and the outer surfaces, and the inner and the outer radii of the cylinder. When $g(r) = \text{constant } a$, then

$$a = (p_{\text{in}} - p_{\text{ou}})/\ln(r_{\text{ou}}/r_{\text{in}}), \quad (17)$$

and through-the-thickness uniform in-plane shear stress, a , is determined by pressures applied on the inner and the outer surfaces and their radii. Thus a can be made small by either decreasing r_{in} or increasing r_{ou} .

When the hoop stress is a prescribed smooth function of r , that is,

$$\sigma_{\theta\theta}(r) = h(r), \quad (18)$$

then integration of Equation (1) with respect to r and boundary conditions (2a, b) gives

$$(-r_{\text{ou}} p_{\text{ou}} + r_{\text{in}} p_{\text{in}}) = \int_{r_{\text{in}}}^{r_{\text{ou}}} h(r) dr. \quad (19)$$

Thus $h(r)$ cannot be an arbitrary function of r , but must satisfy Equation (19). For $h(r) = \text{constant } b$, we get from Equation (19):

$$b = (-r_{\text{ou}} p_{\text{ou}} + r_{\text{in}} p_{\text{in}})/(r_{\text{ou}} - r_{\text{in}}), \quad (20)$$

which is basically the same as Equation (13).

3.2. Sphere

We write Equation (6) as

$$(r^2 \sigma_{rr})' - 2r \sigma_{\theta\theta} = 0, \quad (21)$$

which upon integration with respect to r and using boundary conditions (7a, b) gives

$$-r_{\text{ou}}^2 p_{\text{ou}} + r_{\text{in}}^2 p_{\text{in}} = \sigma_{\theta\theta}^a (r_{\text{ou}}^2 - r_{\text{in}}^2), \quad (22)$$

where $\sigma_{\theta\theta}^a$ equals the constant circumferential stress in the sphere taken to be equal to the average hoop stress. Thus the average hoop stress over the sphere thickness is determined by pressures applied on the inner and the outer surfaces of the sphere and their radii. For $r_{\text{ou}}^2 p_{\text{ou}} = r_{\text{in}}^2 p_{\text{in}}$, the average hoop stress vanishes, and the hoop stress cannot be of the same sign (i.e., either tensile or compressive) through the sphere thickness. *Thus for $p_{\text{in}} = p_{\text{ou}} = 0$, if the residual stresses are functions of r only, then along a radial line the residual hoop stress must be compressive at some points and tensile at other points.* For a solid sphere, $r_{\text{in}} = 0$, and Equation (22) gives

$$\sigma_{\theta\theta}^a = -p_{\text{ou}}. \quad (23)$$

Thus the average hoop stress in a solid sphere with the outer surface traction free must vanish.

In order for Equation (15) to hold for a sphere, we deduce from Eqs. (6) and (15) the following constraint equation for $g(r)$:

$$-p_{\text{ou}} + p_{\text{in}} = 2 \int_{r_{\text{in}}}^{r_{\text{ou}}} \frac{g(r)}{r} dr. \quad (24)$$

Thus for $g(r)$ to be constant a :

$$a = (p_{\text{in}} - p_{\text{ou}})/2 \ln(r_{\text{ou}}/r_{\text{in}}). \quad (25)$$

The constant can be made small by either decreasing the value of r_{in} or increasing r_{ou} .

When $\sigma_{\theta\theta} = \sigma_{\phi\phi} = h(r)$ is desired, then for Eqs. (6) and (7a, b) to hold we must have

$$-r_{\text{ou}}^2 p_{\text{ou}} + r_{\text{in}}^2 p_{\text{in}} = 2 \int_{r_{\text{in}}}^{r_{\text{ou}}} r h(r) dr. \quad (26)$$

For $h(r) = \text{constant } b$, Equation (26) gives

$$b = \frac{-r_{\text{ou}}^2 p_{\text{ou}} + r_{\text{in}}^2 p_{\text{in}}}{r_{\text{ou}}^2 - r_{\text{in}}^2}, \quad (27)$$

which is the same as Equation (22).

We note that Eqs. (12)–(27) are valid for all materials and all types of static axisymmetric deformations, whether or not they are isochoric; *thus they are universal relations*. In addition to $g(r)$ and $h(r)$ satisfying Eqs. (16) and (19), respectively, for a cylinder, they must be such that the corresponding shear modulus found below is positive everywhere; a similar comment applies for a sphere.

The procedure used above to derive the universal relations is similar to that given in Truesdell and Toupin [21], who have given an expression, due to Signorini, for the mean stress in terms of the hydrostatic pressures acting on the inner and the outer surfaces of a body with a cavity, the volume of the cavity, and the volume of the hollow body. This theorem applies to a hollow sphere with pressures acting on the inner and the outer surfaces. Their result (Equation (220.4) of Truesdell and Toupin [21]) shows that the mean stress tensor in such a body is a hydrostatic pressure.

4. Material tailoring

4.1. Cylinder

4.1.1. Cylinder made of incompressible Hookean material

We now find the spatial variation of the shear modulus $\mu(r)$ in Equation (3) for Equations (15) and (18) to hold. For a linear elastic incompressible material, Equations (4) and (5) have the solution

$$u(r) = \frac{\alpha}{r}, \quad (28)$$

where α is a constant. Thus substituting for u from Equation (28) into Equation (4), the result into Equation (3a, b), and then for stresses into Equation (15), we obtain

$$\mu(r) = \frac{r^2}{4\alpha} g(r). \quad (29)$$

In order for $\mu(r) > 0$ for $r_{\text{in}} \leq r \leq r_{\text{ou}}$, $g(r)$ must have the same sign as α for $r_{\text{in}} \leq r \leq r_{\text{ou}}$. For the in-plane shear stress to be constant, a , the shear modulus must be proportional to r^2 , and

$$\mu(r) = \frac{r^2}{4\alpha} (p_{\text{in}} - p_{\text{ou}}) / \ln(r_{\text{ou}}/r_{\text{in}}), \quad (30)$$

where we have used Equation (17). For a known function $g(r)$:

$$\alpha = \frac{r_{\text{in}}^2 g(r_{\text{in}})}{4\mu(r_{\text{in}})} = \frac{r_{\text{ou}}^2 g(r_{\text{ou}})}{4\mu(r_{\text{ou}})}, \quad (31)$$

is determined by values of g and μ at a point either on the inner surface or on the outer surface of the cylinder.

For $\sigma_{\theta\theta}$ given by Equation (18), we get the following from Eqs. (3) and (28):

$$\sigma_{rr} = h(r) - \frac{4\mu(r)\alpha}{r^2}. \quad (32)$$

Substituting from Equations (32) and (18) into equilibrium Equation (1) and integrating the resulting equation with respect to r , we arrive at

$$\mu(r) = \frac{r}{4\alpha} \left[\frac{4\alpha\mu(r_{\text{in}})}{r_{\text{in}}} + \int_{r_{\text{in}}}^r r h' dr \right]. \quad (33)$$

Thus for $\sigma_{\theta\theta}$ to be constant through the cylinder thickness (i.e., $h' = 0$), $\mu(r)$ must be proportional to r , which was also derived in Batra [2] and Nie and Batra [3] by a different approach. For a very thick cylinder (i.e. $r_{\text{ou}} \gg r_{\text{in}}$) subjected to internal pressure only, Equations (13) and (33) imply that high stresses near the inner surface can be eliminated by varying the shear modulus linearly with r .

4.1.2. Cylinder composed of incompressible neo-Hookean material

For plane strain axisymmetric finite deformations to be isochoric:

$$\frac{r}{R} \frac{dr}{dR} = 1. \quad (34)$$

Thus

$$r^2 = A + R^2, \quad (35)$$

where A is a constant. For radial expansion (contraction) of the cylinder A must be positive (negative). For the in-plane shear stress given by Equation (15), we substitute for r from Equation (35) into Equation (10), for strains B_{rr} and $B_{\theta\theta}$ into Equation (9), and then for stresses into Equation (15), to get

$$C_1(r) = \frac{r^2(r^2 - A)}{2A(2r^2 - A)} g(r). \quad (36)$$

If desired, $C_1(r)$ can be expressed as a function of the radius R in the reference configuration. For the in-plane shear stress to be constant a , the material parameter C_1 should vary through the thickness in the reference configuration as

$$C_1(R) = \frac{(A + R^2)R^2}{2A(A + 2R^2)} a, \quad (37)$$

where a is given by Equation (17). For $A > 0$, $(A + R^2)/(A + 2R^2)$ is a decreasing function of R , and the slope of the curve $C_1(R)$ vs. R^2 decreases with an increase in R . For $R_{ou} \ll \sqrt{A}$, $C_1(R)$ given by Equation (37) is nearly proportional to R^2 .

For $\sigma_{\theta\theta}$ given by Equation (18), Equations (35), (9), (10), and (6) yield

$$\sigma_{rr} = h(r) + 2C_1(r) \left[\frac{R^2}{r^2} - \frac{r^2}{R^2} \right]. \quad (38)$$

Substitution from Equations (38) and (35) into Equation (12) and integration of the resulting equation give

$$C_1(r) = \frac{r(A - r^2)}{A(A - 2r^2)} \left[C_1(r_{in}) \frac{A(A - 2r_{in}^2)}{r_{in}(A - r_{in}^2)} + \frac{1}{2} \int_{r_{in}}^r rh' dr \right]. \quad (39)$$

Thus for the hoop stress to be constant b through the cylinder thickness:

$$C_1(r) = C_1(r_{in}) \frac{A - 2r_{in}^2}{A - 2r^2} \frac{A - r^2}{A - r_{in}^2} \frac{r}{r_{in}}, \quad (40)$$

$$b = \frac{-(\sqrt{A + R_{ou}^2})p_{ou} + (\sqrt{A + R_{in}^2})p_{in}}{\sqrt{A + R_{ou}^2} - \sqrt{A + R_{in}^2}}. \quad (41)$$

Equation (41) follows from Equations (13) (or Equation (20)) and (35).

4.1.3. Cylinder comprised of Mooney–Rivlin material

Following the procedure outlined in Section 4.1.2, we get Eqs. (36) and (39) with $C_1(r)$ replaced by $(C_1(r) - C_2(r))$ for the in-plane shear stress and the hoop stress given by Eqs. (15) and (18), respectively. On the assumption that $C_2(r) = \beta C_1(r)$, where β is a constant, the spatial variations of the material parameters for the Mooney–Rivlin and the neo-Hookean materials are the same in order to achieve the desired spatial distributions of the in-plane shear stress and the hoop stress.

4.2. Sphere

4.2.1. Sphere made of incompressible Hookean material

The volume-preserving condition (8) gives

$$u = \frac{\gamma}{r^2}, \quad (42)$$

where γ is a constant. Proceeding in way similar to that for the hollow cylinder considered in Section 4.1.1 we get

$$\mu(r) = \frac{\gamma}{6} g(r) r^3, \quad (43)$$

$$\mu(r) = \frac{r}{r_{in}} \mu(r_{in}) + \frac{r}{6\gamma} \int_{r_{in}}^r r^2 h' dr, \quad (44)$$

respectively, for Eqs. (15) and (18) to hold. Thus for the in-plane shear stress to be constant through the sphere thickness, $\mu(r)$ must be proportional to r^3 . Similarly, the hoop stress will be uniform throughout the sphere when μ is a linear function of r ; this result was derived by Batra [2] by a different approach. It is clear that the stress concentration around a tiny spherical void at the center of a very thick sphere can be avoided by suitably varying μ through the sphere thickness. The function $h(r)$ in Equation (18) must be such that $\mu(r)$ given by Equation (44) is positive for $r_{in} \leq r \leq r_{ou}$.

4.2.2. Sphere composed of incompressible neo-Hookean material

For axisymmetric finite deformations of a sphere to be isochoric:

$$\frac{r^2}{R^2} \frac{dr}{dR} = 1. \quad (45)$$

Hence

$$r^3 = D + R^3, \quad (46)$$

where D is a constant. For radial expansion (contraction) of the sphere, D must be positive (negative). Following the procedure of Section 4.1.2, we conclude that $C_1(r)$ is given by

$$C_1(r) = \frac{1}{2} \frac{r^2(r^3 - D)^{4/3}}{D(2r^3 - D)} g(r), \quad (47)$$

and

$$C_1(r) = \frac{r^2(r^3 - D)^{2/3}}{D(2r^3 - D)} \left[C_1(r_{\text{in}}) D \frac{(2r_{\text{in}}^3 - D)}{r_{\text{in}}^2(r_{\text{in}}^3 - D)^{2/3}} + \frac{1}{2} \int_{r_{\text{in}}}^r r^2 h' dr \right], \quad (48)$$

respectively, for the in-plane shear stress and the hoop stress to be given by Eqs. (15) and (18).

4.2.3. Sphere comprised of Mooney–Rivlin material

As for the cylinder problem studied in Section 4.1.3 for the in-plane shear stress and the hoop stress given by Eqs. (15) and (18), Eqs. (47) and (48) hold with $C_1(r)$ replaced by $(1 - \beta)C_1(r)$, where we have tacitly assumed that $C_2(r) = \beta C_1(r)$.

5. Remarks

For plane strain problems the axial stress in a cylinder is generally non-zero. For linear elastic materials one can use the principle of superposition to find total stresses and deformations due to pressures on the inner and the outer surfaces of the cylinder and axial tractions at the end faces. In addition, the St. Venant principle ensures that different distributions of tractions at the end faces with the same resultant force and the same resultant moment will not affect stresses at points sufficiently far away from the end faces. Thus one can take the axial tractions at the end faces to be uniformly distributed. For a non-linear problem, one cannot use the principle of superposition and the range of validity of the St. Venant principle is unknown, for example, see Horgan and Knowles [22, 23], Berdichevskii [24], Roseman [25], and Breuer and Roseman [26]. Thus for deformations of a hollow cylinder subjected to pressures applied to the inner and the outer surfaces to be axisymmetric, one must apply axial tractions on the end faces as required by the constitutive relation.

The through-the-thickness variations of the material elasticities that are needed to achieve desired radial distributions of stresses depend upon the constitutive relation. One can follow the procedure employed in this paper to analyze similar problems for other material models, for example, biological tissue, the Ogden material, and the Gent material; constitutive relations for these materials are given by Batra [27]. One should note that for the cylinder and the sphere subjected to uniform pressures on the inner and the outer surfaces, there is only one equilibrium equation to be satisfied. Thus we can derive only one differential or algebraic equation for the elasticities. Should the constitutive relation for a material involve more than one elastic parameter (e.g., the Mooney–Rivlin material has two elasticities), then there are several combinations of the spatial variations of the two parameters to achieve the desired variation of either the hoop or the in-plane shear stress.

Problems studied herein have been analyzed without finding an explicit expression for the hydrostatic pressure appearing in the constitutive relations. A more challenging problem is that of ascertaining the spatial variation of material elasticities so as to achieve the desired radial variation of the von Mises stress or another non-linear function of stresses. For these problems one will need to solve the equilibrium equations for the

hydrostatic pressure, and substitute for stresses in the non-linear function. The resulting non-linear equation for the elasticities may not have a unique solution.

We note that nonlinear elastic problems for functionally graded cylinders have been analyzed by Batra and Bahrami [28], and Batra and Iaccarino [29]. During the analysis of FG cylinders, many researchers have assumed that Poisson's ratio is a constant. However, Mohammadi and Dryden [30] as well as Nie and Batra [4] have delineated the effect of varying Poisson's ratio on stresses induced in a cylinder.

6. Conclusions

For axisymmetric deformations, we have derived universal relations for the average in-plane shear stress and the average hoop stress in terms of pressures applied on the inner and the outer surfaces of a cylinder and a sphere and their radii. These relations are valid for both infinitesimal and finite elastic and inelastic deformations, as well as for compressible and incompressible materials. Thus they can be used to check the accuracy of approximate solutions obtained through numerical techniques.

We have also determined the through-the-thickness variation of material parameters (the shear modulus for a Hookean material, a material parameter for a neo-Hookean material, and two material parameters for a Mooney–Rivlin material on the assumption that one is a scalar multiple of the other), so that either the in-plane shear stress or the hoop stress has the desired through-the-thickness variation in a cylinder and a sphere. The variations with respect to the radius r of these stresses cannot be arbitrarily assigned, but must satisfy the constraint equations derived in the paper. The requirement that these stresses be uniform through the cylinder (or the sphere) thickness provides a challenge to material engineers to achieve the needed spatial variations of material parameters. The uniform hoop stress through the cylinder (or sphere) thickness will eliminate stress concentrations near the inner surface in a thick cylinder (or sphere).

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Conflict of interest

None declared.

Dedication

Dedicated to Professor Michael Carroll on his 75th birthday.

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