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ROLLING/SLIDING OF A LINEAR ELASTIC BODY ON A DEFORMABLE LAYERED SUBSTRATE

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Abstract—We analyze a three-dimensional frictional contact problem between a homogeneous and isotropic linear elastic body and a substrate consisting of a homogeneous and isotropic linear elastic half-space coated with a transversely isotropic elastic layer. The problem formulation includes the kinematics of the body, including its rolling/sliding on the substrate. Expressions relating the displacements of a point on the contact surface to the surface tractions acting there have been derived. We have also proposed a variational problem for the determination of surface tractions at points of the contact surface. An example problem involving the motion of a homogeneous spherical rigid ball on a substrate made of a homogeneous thin layer of a linear elastic material and bonded to a rigid base is studied. It is shown that there can be at most one adhesion zone followed by a slipping zone. Numerical results illustrating the dependence of the slip velocity upon the resultant frictional force are also presented graphically.

1. INTRODUCTION

In order to reduce the frictional force between two sliding bodies one often uses solid lubricants such as graphite or Molybdenum disulfide. The outer surface of one of the contacting bodies is usually coated with one of these lubricants. The mechanical properties of these coatings in the thickness direction are different from those in the plane of coating. It seems reasonable to model the coating as a transversely isotropic layer. In many applications, e.g. rolling bearings, gears and railway vehicles, sliding speeds are much smaller than the magnitude of the characteristic linear velocity of the body. However, sliding speeds are not negligible, thereby suggesting that the kinematics of contacting bodies corresponds to rolling with sliding.

Problems involving the smooth contact of a rigid body with a layered substrate have been studied by Hannah [1], Aleksandrov [2], Meijers [3] and Alblas and Kuipers [4]. Frictionless contact of an elastic body with a layered base has been investigated by Gupta and Walowit [5], O'Sullivan and King [6], Komvopoulos [7] and Komvopoulos *et al.* [8] have considered frictional problems involving the pure sliding of one body over the other. Sliding contact of an indenter on a transversely isotropic layer has been studied by Kuo and Keer [9]. Kalker [10] has examined the rolling/sliding of a body on an elastic half-space, and Bhargava *et al.* [11] have developed elastic-plastic models for rolling contact problems.

Here we consider the rolling with sliding of a body on an elastic base coated with a transversely isotropic layer. Slipping takes place on an *a priori* unknown part of the contact area. Inertial properties of the body assumed as rigid and elastic deformations of the body points and of the base in the vicinity of the contact area are accounted for. Contact stresses influence the kinematics of the body through its equations of motion. If the adhesion and inertia effects are neglected, then our problem reduces to one of pure quasistatic sliding, studied in refs [6] and [9].

An iterative method of solution of the contact problem is proposed. In every iteration, one needs to solve a variational problem in which the functionals depend upon the pressure and frictional forces acting on the contact area. The contact area, slip and adhesion subareas and surface tractions are determined simultaneously from the solution of a variational problem. The problem of the rolling of a rigid ball on an elastic transversely isotropic

layer bonded to a rigid half-space is studied in detail and numerical results are presented graphically.

2. GOVERNING EQUATIONS

We study the motion of an elastic body on a transversely isotropic layer $0 \leq Z \leq h$ bonded to a half-space occupying the domain $Z < 0$ (cf. Fig. 1). The coordinate axes $0\bar{x}\bar{y}\bar{z}$ with origin at the center of mass of the body coincide with the principal axes of inertia of the undeformed body and are embedded in it. The coordinate axes $0xyz$ move with the body but always stay parallel to the fixed XYZ axes. We assume that the contact surface S_c lies in the plane $Z = h$ and its size is small as compared to a typical dimension of the body. At points of S_c we have

$$p = 0 \quad \text{if } Z^+ - Z^- > 0, \quad (2.1)$$

$$p \geq 0 \quad \text{if } Z^+ - Z^- = 0 \quad (2.2)$$

and

$$|\tau| \leq \mu p \quad \text{if } |\mathbf{s}| = 0, \quad (2.3)$$

$$\tau = \mu p \frac{\mathbf{s}}{|\mathbf{s}|} \quad \text{if } |\mathbf{s}| > 0. \quad (2.4)$$

Here p is the contact pressure, τ the tangential traction, μ the coefficient of friction, $\mathbf{s}$ the local slip velocity and Z^+ and Z^- are, respectively, the Z -coordinates of points of the body and the substrate that have the same values of X - and Y -coordinates. $Z^\pm$ -coordinates depend upon the position of the body and on the elastic deformations of the half-space, coating and points of the body in the vicinity of the contact area. In (2.3) and (2.4) we have assumed Coulomb's law of friction. Condition (2.1) holds at points of S_c where the two bodies are not in contact with each other and on the remainder of S_c the normal traction is compressive and the tangential tractions are given by (2.4) and (2.3), according as to whether there is or is not slipping between the two bodies.

The slip velocity $\mathbf{s}$ defined at points of S_c can be written as

$$\mathbf{s} = \mathbf{v} + \frac{d}{dt}(\mathbf{u}^+ - \mathbf{u}^-), \quad (2.5)$$

where $\mathbf{v}$ equals the slip velocity when the body and the base are regarded as absolutely rigid and $\mathbf{u}^+$ and $\mathbf{u}^-$ are the tangential displacements, with respect to the fixed coordinate axes, of the body and the base due to their elastic deformations. In linear elasticity, one does not distinguish between the material time derivative and the partial time derivative because of the infinitesimal deformations involved. However, for the present problem, the speed of a material point of the body can be quite large as compared to the rate of change of its

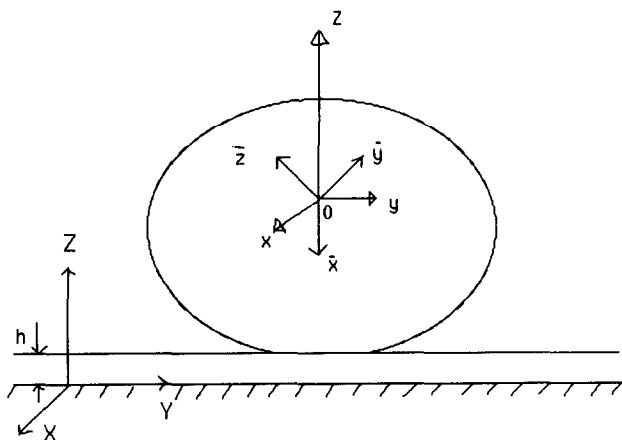


Fig. 1. Schematic sketch of the problem studied.

elastic deformations. Thus,

$$\mathbf{s} = \mathbf{v} + V_x \frac{\partial}{\partial x}(\mathbf{u}^+ - \mathbf{u}^-) + V_y \frac{\partial}{\partial y}(\mathbf{u}^+ - \mathbf{u}^-), \quad (2.6)$$

where V_x and V_y are the x - and y -components of the velocity of a point on the contact surface. The term involving V_z is omitted since $|V_z| \ll |V_x|$ and $|V_y|$. For rolling/sliding contact, $V_x \simeq -\dot{R}_x$, $V_y \simeq -\dot{R}_y$, where R_x and R_y are the x - and y -components of the position vector of the center of mass of the body. Equations that determine $\mathbf{R}$ are

$$M \ddot{\mathbf{R}} = \mathbf{F} + \int_{S_c} \boldsymbol{\sigma}^{sb} dA, \quad (2.7)$$

$$\mathbf{I} \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathbf{I} \boldsymbol{\Omega} = \mathbf{T} + \int_{S_c} \mathbf{r} \times \boldsymbol{\sigma}^{sb} dA. \quad (2.8)$$

Here M is the mass of the body, $\mathbf{F}$ and $\mathbf{T}$ are resultant forces and moments due to external forces (except those on S_c) applied to the body, $\boldsymbol{\sigma}^{sb}$ is the surface traction acting on the body at its points on the contact surface S_c , $\boldsymbol{\Omega}$ is its angular velocity and $\mathbf{I}$ is the inertia tensor with respect to the principal axes $0\bar{x}\bar{y}\bar{z}$ of inertia embedded in the body. Equation (2.7) describes the motion of the center of mass of the body and equation (2.8) is the evolution equation for the angular velocity $\boldsymbol{\Omega}$ of the body assumed as rigid.

3. FORMULATION OF BOUNDARY CONDITIONS IN TERMS OF CONTACT STRESSES

Since S_c is very small as compared to the size of the body, we can determine elastic displacements of body points in the vicinity of S_c by approximating it locally as a half-space (e.g. see [10]) and using the Boussinesq–Cerruti's formulae [10]. Thus,

$$w^+ = B_{11}^+(p) + B_{12}^+(\boldsymbol{\tau}), \quad (3.1)$$

$$\mathbf{u}^+ = \mathbf{B}_{21}^+(p) + \mathbf{B}_{22}^+(\boldsymbol{\tau}), \quad (3.2)$$

where

$$B_{11}^+(p) = \frac{1 - \nu^+}{2\pi G^+} \int_{S_c} \frac{p}{\hat{r}} dx' dy', \quad (3.3)$$

$$B_{12}^+(\boldsymbol{\tau}) = -\alpha \int_{S_c} \frac{1}{\hat{r}} (\tau_{x'z} \cos \theta + \tau_{y'z} \sin \theta) dx' dy', \quad (3.4)$$

$$\mathbf{B}_{21}^+(p) = -\alpha \int_{S_c} \frac{1}{\hat{r}} \begin{Bmatrix} \cos \theta \\ \sin \theta \end{Bmatrix} p dx' dy', \quad (3.5)$$

$$\mathbf{B}_{22}^+(\boldsymbol{\tau}) = -\frac{1}{2\pi G^+} \int_{S_c} \frac{1}{\hat{r}} \begin{Bmatrix} (1 - \nu^+ \sin^2 \theta) \tau_{x'z} + \nu^+ \sin \theta \cos \theta \tau_{y'z} \\ \nu^+ \sin \theta \cos \theta \tau_{y'z} + (1 - \nu^+) \cos^2 \theta \tau_{x'z} \end{Bmatrix} dx' dy', \quad (3.6)$$

$$\alpha = \frac{1 - 2\nu^+}{4\pi G^+}, \quad \hat{r} = [(x - x')^2 + (y - y')^2]^{1/2}, \quad (3.7)$$

$$\cos \theta = \frac{x - x'}{\hat{r}}, \quad \sin \theta = \frac{y - y'}{\hat{r}}, \quad (3.8)$$

G^+ and ν^+ equal, respectively, the shear modulus and Poisson's ratio for the material of the body.

The displacements at a point of a transversely isotropic layer with elasticities c_{11} , c_{12} , c_{13} , c_{33} and c_{44} can be expressed in terms of three potential functions as [12]

$$u_x^- = (\phi_1 + \phi_2)_{,xz} + \psi_{,y}, \quad (3.9)$$

$$u_y^- = (\phi_1 + \phi_2)_{,yz} + \psi_{,x}, \quad (3.10)$$

$$u_z^- = w^- = -[c_{11}((\phi_1 + \phi_2)_{,xx} + (\phi_1 + \phi_2)_{,yy}) + c_{44}(\phi_1 + \phi_2)_{,zz}]/(c_{13} + c_{44}). \quad (3.11)$$

Here a comma followed by x implies partial differentiation with respect to x , and functions

ϕ_1 , ϕ_2 , and ψ satisfy the following equations:

$$\phi_{1,xx} + \phi_{1,yy} + \frac{1}{v_1^2} \phi_{1,zz} = 0, \quad (3.12)$$

$$\phi_{2,xx} + \phi_{2,yy} + \frac{1}{v_2^2} \phi_{2,zz} = 0, \quad (3.13)$$

$$\psi_{,xx} + \psi_{,yy} + \frac{1}{v_3^2} \psi_{,zz} = 0, \quad (3.14)$$

where

$$v_1 = \left[\frac{1}{4c_{33}c_{44}}(d_{13} - c_{13})(d_{13} + c_{13} + 2c_{44}) \right]^{1/2} + \left[\frac{1}{4c_{33}c_{44}}(d_{13} + c_{13})(d_{13} - c_{13} - 2c_{44}) \right]^{1/2}, \quad (3.15)$$

$$v_2 = \left[\frac{1}{4c_{33}c_{44}}(d_{13} - c_{13})(d_{13} + c_{13} + 2c_{44}) \right]^{1/2} - \left[\frac{1}{4c_{33}c_{44}}(d_{13} + c_{13})(d_{13} - c_{13} - 2c_{44}) \right]^{1/2}, \quad (3.16)$$

$$v_3 = (c_{33}/c_{44})^{1/2}, \quad d_{13} = (c_{11}c_{13})^{1/2}. \quad (3.17)$$

Taking the Fourier transform of each term in equations (3.12)–(3.14) with respect to arguments x and y , we obtain

$$\tilde{\phi}_i = A_i e^{v_i z \delta} + B_i e^{-v_i z \delta}, \quad i = 1, 2, \quad \delta = \sqrt{\xi^2 + \eta^2}, \quad (3.18)$$

$$\tilde{\psi} = A_3 e^{v_3 z \delta} + B_3 e^{-v_3 z \delta}, \quad (3.19)$$

where $\tilde{\psi}$ signifies the Fourier transform of ψ with parameters ξ, η , and parameters $\mathbf{A} \equiv (A_1, A_2, A_3, B_1, B_2, B_3)$ are constants.

It is convenient to introduce vectors $\tilde{\mathbf{W}}$ and $\tilde{\mathbf{F}}$ by the formulae

$$\tilde{\mathbf{W}} = \left[\frac{i}{\delta^3} (\xi \tilde{u}_x^- + \eta \tilde{u}_y^-), \frac{i}{\delta^2} (\eta \tilde{u}_x^- - \xi \tilde{u}_y^-), -\frac{\tilde{u}_z^-}{\delta^2} \right], \quad (3.20)$$

$$\tilde{\mathbf{F}} = \left[\frac{i}{c_{44}\delta^4} (\xi \tilde{\sigma}_{xz} + \eta \tilde{\sigma}_{yz}), \frac{i}{c_{44}v_3\delta^3} (\eta \tilde{\sigma}_{xz} - \xi \tilde{\sigma}_{yz}), \frac{\tilde{\sigma}_{zz}}{c_{13}\delta^3} \right]. \quad (3.21)$$

Taking the Fourier transform of each term in equations (3.9)–(3.11), and using (3.18) and (3.19), we obtain

$$[\mathbf{C}(\xi, \eta, z)] \mathbf{A} = \begin{Bmatrix} \tilde{\mathbf{W}}(z) \\ \tilde{\mathbf{F}}(z) \end{Bmatrix}. \quad (3.22)$$

Equation (3.22) enables one to express displacements and stresses at the bottom surface $Z = 0$ of the coating in terms of those at its top surface $Z = h$, i.e.

$$\begin{Bmatrix} \tilde{\mathbf{W}}(0) \\ \tilde{\mathbf{F}}(0) \end{Bmatrix} = [\mathbf{C}(\xi, \eta, 0)] [\mathbf{C}^{-1}(\xi, \eta, h)] \begin{Bmatrix} \tilde{\mathbf{W}}(h) \\ \tilde{\mathbf{F}}(h) \end{Bmatrix}, \quad (3.23)$$

where elements of matrix $[\mathbf{D}] = [\mathbf{C}(\xi, \eta, 0)] [\mathbf{C}^{-1}(\xi, \eta, h)]$ are defined in the appendix. Due to adhesion conditions at the interface between the coating and the supporting half-space, we have

$$\tilde{\mathbf{W}}(0) = \tilde{\mathbf{W}}^s(0), \quad \tilde{\mathbf{F}}(0) = \tilde{\mathbf{F}}^s(0), \quad (3.24)$$

where the superscript s signifies a quantity for the half-space. According to the Bousinesq–Cerruti's [10] formulae,

$$\tilde{\mathbf{W}}^s(0) = \tilde{\mathbf{B}}(\xi, \eta) \tilde{\mathbf{F}}^s(0), \quad (3.25)$$

where the elements of the matrix $\tilde{\mathbf{B}}$ are combinations of Fourier transforms of B_{ij} defined by

equations (3.3)–(3.6). We rewrite (3.23) as

$$\tilde{\mathbf{W}}(0) = \mathbf{D}_{11}(\xi, \eta) \tilde{\mathbf{W}}(h) + \mathbf{D}_{12}(\xi, \eta) \tilde{\mathbf{F}}(h), \quad (3.26)$$

$$\tilde{\mathbf{F}}(0) = \mathbf{D}_{21}(\xi, \eta) \tilde{\mathbf{W}}(h) + \mathbf{D}_{22}(\xi, \eta) \tilde{\mathbf{F}}(h), \quad (3.27)$$

where matrices $\mathbf{D}_{11}$, $\mathbf{D}_{12}$, $\mathbf{D}_{21}$ and $\mathbf{D}_{22}$ are determined by matrices $\mathbf{C}$ and $\mathbf{C}^{-1}$. We can combine (3.25)–(3.27) to arrive at

$$\tilde{\mathbf{W}}(h) = (\mathbf{D}_{11} - \tilde{\mathbf{B}}\mathbf{D}_{21})^{-1}(\tilde{\mathbf{B}}\mathbf{D}_{22} - \mathbf{D}_{12})\tilde{\mathbf{F}}(h). \quad (3.28)$$

Hence, surface displacements of the coating/layer can be expressed in terms of surface tractions $\boldsymbol{\sigma}^{\text{sc}} = -\boldsymbol{\sigma}^{\text{sb}}$ at the contact surface in the form

$$\mathbf{w}^- = \mathbf{B}_{11}^-(p) + \mathbf{B}_{12}^-(\boldsymbol{\tau}), \quad (3.29)$$

$$\mathbf{u}^- = \mathbf{B}_{21}^-(p) + \mathbf{B}_{22}^-(\boldsymbol{\tau}), \quad (3.30)$$

where the operators $\mathbf{B}_{11}^-$, $\mathbf{B}_{12}^-$, $\mathbf{B}_{21}^-$ and $\mathbf{B}_{22}^-$ are determined by D_{11} , D_{21} , etc.

4. VARIATIONAL FORMULATION OF THE CONTACT PROBLEM

We consider the system of equations (2.1)–(2.4) at any instant of time and therefore assume that the position vector $\mathbf{R}$ of the center of mass of the body and the relative orientation of principal axes $0\bar{x}\bar{y}\bar{z}$ of inertia are known. We consider the following iterative procedure for the solution of the problem defined by (2.1)–(2.4):

$$p_k = 0 \quad \text{if } Z^+ - Z^- \equiv [Z(p_k, \boldsymbol{\tau}_{k-1})] > 0, \quad (4.1)$$

$$p_k \geq 0 \quad \text{if } [Z(p_k, \boldsymbol{\tau}_{k-1})] = 0 \quad (4.2)$$

and

$$|\boldsymbol{\tau}_k| \leq \mu p_k \quad \text{if } |\mathbf{s}(p_k, \boldsymbol{\tau}_k)| = 0, \quad (4.3)$$

$$\boldsymbol{\tau}_k = \mu p_k \frac{\mathbf{s}(p_k, \boldsymbol{\tau}_k)}{|\mathbf{s}(p_k, \boldsymbol{\tau}_k)|} \quad \text{if } |\mathbf{s}(p_k, \boldsymbol{\tau}_k)| > 0. \quad (4.4)$$

Surface tractions for $k = 0$ are presumed to be known. The major idea of this process is to have the problem of the determination of the pressure and contact area uncoupled from each other. This is reasonable for contact problems in which the coefficient of friction is not very large, since pressure and contact area depend rather weakly upon friction forces. Iterative processes of this type have been used in static [13, 14] and stationary [10] problems.

We prove below that each one of the problems defined by equations (4.1) and (4.2), and (4.3) and (4.4), is equivalent to a variational problem. We first consider the problem defined by equations (4.1) and (4.2) and write the function $[Z]$ in the form

$$[Z(p_k, \boldsymbol{\tau}_{k-1})] = z + R_z - B_{11}(p_k) + B_{12}(\boldsymbol{\tau}_{k-1}), \quad (4.5)$$

where

$$B_{11}(p_k) = B_{11}^+(p_k) - B_{11}^-(p_k), \quad B_{12}(\boldsymbol{\tau}_{k-1}) = B_{12}^+(\boldsymbol{\tau}_{k-1}) - B_{12}^-(\boldsymbol{\tau}_{k-1}) \quad (4.6)$$

and $z = z(x, y, t)$ defines the surface S_c . We assert that the variational problem of minimizing $F(p)$, where

$$F(p) \equiv \int_{S_c} [\tfrac{1}{2} B_{11}(p)p + fp] dA, \quad (4.7)$$

$$f = -z - R_z - B_{12}(\boldsymbol{\tau}_{k-1}), \quad p \geq 0, \quad (4.8)$$

is equivalent to solving equations (4.1) and (4.2) for p_k . We first prove that (4.7) and (4.8) imply (4.1) and (4.2). The quadratic part of the functional $F(p)$ equals one-half the elastic energy of the body and the base (the half-space and coating) corresponding to surface tractions $\sigma_{zz} = -p$, $\tau_{xz} = \tau_{yz} = 0$ on S_c . Thus,

$$\int_{S_c} B_{11}(p)p dA \geq 0 \quad (4.9)$$

and the functional $F(p)$ is convex. Hence, the problem of minimizing (4.7) is equivalent to

finding p_0 , such that

$$\int_{S_c} (B_{11}(p_0) + f)(p - p_0) dA \geq 0 \quad \forall p > 0. \quad (4.10)$$

It implies that the linear part of the increment of the functional $F(p)$ should be non-negative if $F(p)$ attains its minimum value for $p = p_0$. Let p_0 be the solution of the aforestated variational problem. We can write it as

$$l(p_0) = \inf_{p \geq 0} l(p), \quad (4.11)$$

where

$$l(p) = \int_{S_c} (B_{11}(p_0) + f)p dA. \quad (4.12)$$

Since $l(0) = 0$, we have $l(p_0) \leq 0$. Assume that $l(p_0) < 0$. Then for $p = \lambda p_0$, $\lambda > 1$, $l(p) < l(p_0)$, which contradicts (4.12). Hence,

$$l(p_0) = 0 \quad (4.13)$$

and

$$\int_{S_c} [Z^+ - Z^-](p_0)p dA \geq 0 \quad \forall p \geq 0, \quad (4.14)$$

which is an integral form of boundary conditions (4.1) and (4.2). If functions p_0 and $[Z^+ - Z^-](p_0)$ are continuous, conditions (4.1) and (4.2) can be obtained in the local form.

In order to prove the converse, let p_0 be a solution of the problem defined by equations (4.1) and (4.2). Thus,

$$B_{11}(p_0) + f \geq 0 \quad (4.15)$$

and

$$[B_{11}(p_0) + f]p_0 \equiv 0. \quad (4.16)$$

Therefore, inequality (4.10) is satisfied. The regions where conditions (2.1) or (2.2) hold are determined by function p_0 after the solution of the variational problem has been found.

We now consider the problem given by (4.3) and (4.4) for the determination of friction forces. Substitution from (3.2) and (3.30) into (2.5) gives

$$\mathbf{s} = \mathbf{v}^* - \mathbf{B}^*(\boldsymbol{\tau}), \quad (4.17)$$

where

$$\mathbf{v}^* = \mathbf{v} + V_x \frac{\partial}{\partial x} (B_{21}^+(p) - B_{21}^-(p)) + V_y \frac{\partial}{\partial y} (B_{21}^+(p) - B_{21}^-(p)), \quad (4.18)$$

$$\mathbf{B}^*(\boldsymbol{\tau}) = -V_x \frac{\partial}{\partial x} (B_{22}^+(\boldsymbol{\tau}) - B_{22}^-(\boldsymbol{\tau})) - V_y \frac{\partial}{\partial y} (B_{22}^+(\boldsymbol{\tau}) - B_{22}^-(\boldsymbol{\tau})). \quad (4.19)$$

The elements of matrix operators $B_{22}^\pm$ are symmetric with respect to pairs (x, y) and (x', y') . The differentiation of $B_{22}^\pm$ with respect to x and y yields skew-symmetric and singular kernels. Hence,

$$\int_{S_c} \boldsymbol{\tau}^1 \cdot \mathbf{B}^*(\boldsymbol{\tau}^2) dA = - \int_{S_c} \boldsymbol{\tau}^2 \cdot \mathbf{B}^*(\boldsymbol{\tau}^1) dA, \quad (4.20)$$

$$\int_{S_c} \boldsymbol{\tau} \cdot \mathbf{B}^*(\boldsymbol{\tau}) dA = 0. \quad (4.21)$$

Consider the following variational problem. Find $\boldsymbol{\tau}$ such that

$$\mathcal{L}(\boldsymbol{\tau}) \equiv \int_{E_k} [\mu p_k |\mathbf{s}(\boldsymbol{\tau}, p_k)| - \boldsymbol{\tau} \cdot \mathbf{s}(\boldsymbol{\tau}, p_k)] dA, \quad (4.22)$$

where $\mathbf{s}$ is determined by $\mathbf{v}^*$ and $p = p_k$ takes on a minimum value for every $|\boldsymbol{\tau}| \leq \mu p_k$. We assume that the contact area E_k is determined by taking $p = p_k$ and $\boldsymbol{\tau} = \boldsymbol{\tau}_{k-1}$. Due to (4.21), we can rewrite (4.22) as

$$\mathcal{L}(\boldsymbol{\tau}) = \int_{E_k} [\mu p_k |\mathbf{s}(\boldsymbol{\tau}, p_k)| - \boldsymbol{\tau} \cdot \mathbf{v}^*] dA. \quad (4.23)$$

Let τ_0 be a solution of the problem defined by equations (4.3) and (4.4). Then

$$\tau_0 \cdot \mathbf{s}(\tau_0) = \mu p_k |\mathbf{s}(\tau_0)| \quad (4.24)$$

and $\mathcal{L}(\tau_0) = 0$. Since $\mathcal{L}(\tau) \geq 0$, $\mathcal{L}$ takes on its minimum value at a solution of the problem given by (4.3) and (4.4). In order to prove the converse, we regard functions τ and $\mathbf{s}$ as independent variables constrained by

$$\mathbf{s} - \mathbf{v}^* + \mathbf{B}^*(\tau) = \mathbf{0}, \quad (4.25)$$

and introduce the Lagrangian

$$\bar{\mathcal{L}}(\tau, \mathbf{s}, \lambda) = \int_{E_k} [\mu p_k |\mathbf{s}| - \tau \cdot \mathbf{v}^* - \lambda \cdot (\mathbf{s} - \mathbf{v}^* + \mathbf{B}^*(\tau))] dA, \quad (4.26)$$

where λ is a Lagrange multiplier. By varying the argument τ in $\bar{\mathcal{L}}(\tau, \mathbf{s}, \lambda)$, we get the following necessary condition for the minimum of $\bar{\mathcal{L}}(\tau, \mathbf{s}, \lambda)$ at the point $\tau = \tau_0$, $\mathbf{s} = \mathbf{s}_0$:

$$- \int_{E_k} \mathbf{b} \cdot (\tau - \tau_0) dA \geq 0 \quad \forall |\tau| \leq \mu p_k, \quad (4.27)$$

where

$$\mathbf{b} = \mathbf{s}_0 + \mathbf{B}^*(\tau_0 - \lambda). \quad (4.28)$$

We can rewrite (4.27) in the form

$$\sup_{|\tau| \leq \mu p_k} \int_{E_k} \mathbf{b} \cdot \tau dA = \int_{E_k} \mathbf{b} \cdot \tau_0 dA. \quad (4.29)$$

Using results from convex analysis, we get

$$\sup_{|\tau| \leq \mu p_k} \int_{E_k} \mathbf{b} \cdot \tau dA = \int_{E_k} \mu p_k |\mathbf{b}| dA, \quad (4.30)$$

which when combined with (4.29) gives

$$\mu p_k |\mathbf{b}| - \tau_0 \cdot \mathbf{b} \equiv 0. \quad (4.31)$$

If we vary the argument $\mathbf{s}$ in $\bar{\mathcal{L}}(\tau, \mathbf{s}, \lambda)$, we obtain the condition for the minimum of $\bar{\mathcal{L}}$ in the form

$$\int_{E_k} [\mu p_k (|\mathbf{s}| - |\mathbf{s}_0|) - \lambda \cdot (\mathbf{s} - \mathbf{s}_0)] dA \geq 0 \quad (4.32)$$

or

$$\int_{E_k} [\mu p_k |\mathbf{s}| - \lambda \cdot \mathbf{s}] dA - \int_{E_k} [\mu p_k |\mathbf{s}_0| - \lambda \cdot \mathbf{s}_0] dA \geq 0. \quad (4.33)$$

Assume now that, for some $\mathbf{s} = \mathbf{s}^*$, the first term on the left-hand side of the inequality (4.33) is negative, and consider inequality (4.33) for $\mathbf{s} = \omega \mathbf{s}^*$, where $\omega > 0$. We have

$$\omega \int_{E_k} [\mu p_k |\mathbf{s}^*| - \lambda \cdot \mathbf{s}^*] dA - \int_{E_k} [\mu p_k |\mathbf{s}_0| - \lambda \cdot \mathbf{s}_0] dA \geq 0. \quad (4.34)$$

For sufficiently large ω the sign of the left-hand side in (4.34) is determined by the first negative term that contradicts the inequality. Therefore,

$$\int_{E_k} [\mu p_k |\mathbf{s}| - \lambda \cdot \mathbf{s}] dA \geq 0 \quad \forall \mathbf{s}. \quad (4.35)$$

Setting $\mathbf{s} = \lambda$ in (4.35) gives

$$\int_{E_k} |\lambda| (\mu p_k - |\lambda|) dA \geq 0, \quad (4.36)$$

and hence

$$|\lambda| \leq \mu p_k. \quad (4.37)$$

For sufficiently small ω the sign of the left-hand side in (4.35) is governed by the second

term, and

$$\int_{E_k} [\mu p_k |\mathbf{s}_0| - \boldsymbol{\lambda} \cdot \mathbf{s}_0] dA \leq 0, \quad (4.38)$$

which holds only if

$$\int_{E_k} [\mu p_k |\mathbf{s}_0| - \boldsymbol{\lambda} \cdot \mathbf{s}_0] dA = 0. \quad (4.39)$$

Equation (4.39) is equivalent to

$$\mathbf{s}_0 + \mathbf{B}^*(\boldsymbol{\tau}_0 - \boldsymbol{\lambda}) = \alpha \boldsymbol{\tau}_0, \quad \alpha \geq 0, \quad (4.40)$$

$$\alpha(|\boldsymbol{\tau}_0| - \mu p) \equiv 0 \quad \text{in } E_k. \quad (4.41)$$

Consider the quadratic form

$$W(\boldsymbol{\lambda} - \boldsymbol{\tau}_0, \boldsymbol{\lambda} - \boldsymbol{\tau}_0) = - \int_{E_k} (\boldsymbol{\lambda} - \boldsymbol{\tau}_0) \cdot \mathbf{B}^*(\boldsymbol{\lambda} - \boldsymbol{\tau}_0) dA. \quad (4.42)$$

Due to (4.21) we have

$$W(\boldsymbol{\lambda} - \boldsymbol{\tau}_0, \boldsymbol{\lambda} - \boldsymbol{\tau}_0) = 0, \quad (4.43)$$

and substitution from (4.40) into (4.42) yields

$$W(\boldsymbol{\lambda} - \boldsymbol{\tau}_0, \boldsymbol{\lambda} - \boldsymbol{\tau}_0) = \int_{E_k} \alpha(\boldsymbol{\lambda} \cdot \boldsymbol{\tau}_0 - \boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_0) dA + \int_{E_k} (\boldsymbol{\tau}_0 \cdot \mathbf{s}_0 - \boldsymbol{\lambda} \cdot \mathbf{s}_0) dA = 0. \quad (4.44)$$

Recalling that $\alpha \geq 0$ and $\boldsymbol{\lambda}$ is arbitrary, we conclude that

$$\int_{E_k} [\mu p_k |\mathbf{s}_0| - \mathbf{s}_0 \cdot \boldsymbol{\tau}_0] dA = 0,$$

which when combined with the inequality

$$|\boldsymbol{\tau}_0| \leq \mu p_k$$

represents boundary conditions (4.3) and (4.4) in an integral form. For continuous $\boldsymbol{\tau}_0$ and $\mathbf{s}_0$, the integral form implies the local form of conditions (4.3) and (4.4).

The slip and adhesion subareas where (2.4) and (2.3) hold, respectively, are determined by the function $\boldsymbol{\tau}_0$ after the variational problem has been solved.

5. AN EXAMPLE

We study the motion of a rigid spherical ball of radius R on a transversely isotropic layer bonded to a rigid half-space and assume that

$$\varepsilon = h/a \ll 1, \quad (5.1)$$

where a is the radius of the contact area. We further assume that

$$\frac{\partial \mathbf{u}}{\partial x} \ll \frac{\partial \mathbf{u}}{\partial z} \approx \frac{\mathbf{u}(x, y, h)}{h}, \quad \frac{\partial w^h}{\partial x} \ll \frac{\partial w}{\partial z} \approx \frac{w(x, y, h)}{h}. \quad (5.2)$$

By using (5.2) and Hooke's law, we obtain

$$p(x, y) = -\frac{c_{33}}{h} w(x, y, h) = \frac{c_{33}}{2Rh} (a^2 - x^2 - y^2), \quad (5.3)$$

$$\boldsymbol{\tau}(x, y) = \frac{c_{44}}{h} \mathbf{u}(x, y, h), \quad (5.4)$$

$$a = \left(\frac{4RhP}{\pi c_{33}} \right)^{1/4}, \quad (5.5)$$

where P is the total vertical load acting on the contact area. Johnson [15] compared the solution of the form given by (5.3)–(5.5) for the indentation of a rigid cylinder in an isotropic

layer with the numerical results of Meijers [3] and concluded that approximation (5.2) yields satisfactory results if $\varepsilon < 0.5$ and $\nu < 0.45$.

We first assume that a steady state has been reached and

$$v_x^0 = (V - \Omega R)/V, \quad v_y^0 = 0 \quad (5.6)$$

do not vary with time. Subsequently, we will study the evolution problem. There are two possibilities; either v_x^0 is known and the frictional forces are to be determined, or the resultant frictional force is known and v_x^0 should be found. For solutions of the form (5.4), τ_{xz} and τ_{yz} are uncoupled. Here we seek a solution of the problem for which $\tau_{yz} \equiv 0$, $s_y = 0$, and set $\tau = \tau_{xz}$ and $s = s_x$. From (2.6) and (5.4) we obtain

$$s = v_x^0 + \frac{Vh}{c_{44}} \frac{\partial \tau}{\partial x}. \quad (5.7)$$

Thus, we can solve the problem on a line $y = \text{constant}$. We now determine the slip and adhesion zones on this line.

Due to (5.7) τ should be a linear function of x within the adhesion zones and according to (5.3) it should be a parabolic function of x within the slip area. Since $\partial \tau / \partial x = \text{constant}$ in all possible adhesion zones [see Fig. 2(a)] and a parabolic function cannot be conjugated continuously with different affine functions having the same slope [Fig. 2(b)], we conclude that there can be at most one adhesion zone. In order to determine its location on the line $y = y_0$, we first assume that it is situated between the front and back slip zones. Since

$$k_\ell \equiv \frac{\partial \tau}{\partial x}(x_0) = \frac{\mu c_{33}}{Rh} x_0, \quad (5.8)$$

the slip condition at the edge $x_0(y_0)$ of the front slip zone can be written as

$$v_x^0 - \frac{V\mu c_{33}}{c_{44}R} x_0 > 0. \quad (5.9)$$

Inequality (5.9) asserts that the slope

$$k_a = \frac{v_x^0 c_{44}}{Vh} \quad (5.10)$$

of the affine distribution of τ inside the adhesion zone [Fig. 2(c)] exceeds the limiting value k_ℓ . Therefore, only one front adhesion zone and one back slip zone are possible inside every segment of the contact line $y = y_0$ [Fig. 2(d)]. The distribution of the frictional force on this line is given by

$$\tau(x, y) = \begin{cases} \frac{\mu c_{33}}{2Rh} (a^2 - x^2 - y^2) & \text{if } -(a^2 - y^2)^{1/2} \leq x \leq l_a, \\ -\frac{v_x^0 c_{44}}{V} \frac{1}{h} (x - (a^2 - y^2)^{1/2}) & \text{if } l_a \leq x \leq (a^2 - y^2)^{1/2}, \end{cases} \quad (5.11)$$

$$\tau(x, y) = \begin{cases} \frac{\mu c_{33}}{2Rh} (a^2 - x^2 - y^2) & \text{if } -(a^2 - y^2)^{1/2} \leq x \leq l_a, \\ -\frac{v_x^0 c_{44}}{V} \frac{1}{h} (x - (a^2 - y^2)^{1/2}) & \text{if } l_a \leq x \leq (a^2 - y^2)^{1/2}, \end{cases} \quad (5.12)$$

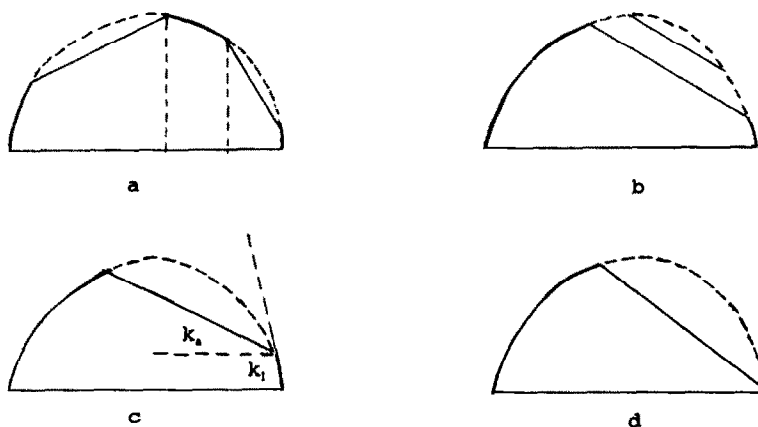


Fig. 2. Different possible locations of the adhesion and slip zones.

where

$$\frac{l_a}{a} = \frac{Bv_x^0}{V} - \left(1 - \left(\frac{y}{a}\right)^2\right)^{1/2}, \quad B = \frac{2Rc_{44}}{\mu ac_{33}}. \quad (5.13)$$

The dimensionless parameter B being proportional to c_{44}/c_{33} is a measure of the anisotropy of the material of the layer. For an isotropic material

$$\frac{c_{44}}{c_{33}} = \frac{1 - 2\nu}{2(2 - \nu)}. \quad (5.14)$$

The value of c_{44}/c_{33} for different solid lubricants can be determined from the data given in [16].

The magnitude of the maximum frictional force within the contact area can be found from

$$\tau_{\max} = \begin{cases} \frac{c_{44}av_x^0}{hV} \left(2 - B\frac{v_x^0}{V}\right) & \text{if } \frac{Bv_x^0}{V} < 1, \\ \frac{\mu c_{33}a^2}{2Rh} & \text{if } \frac{Bv_x^0}{V} \geq 1, \end{cases} \quad (5.15)$$

$$(5.16)$$

and the total frictional force T is given by

$$T = \frac{\mu c_{33}a^4}{Rh} [0.25\pi + \beta y^*(13 + 0.5\beta^2)/12 - 0.5(1 + \beta^2)\arcsin y^*], \quad (5.17)$$

where

$$\beta = \frac{Bv_x^0}{V}, \quad y^* = (1 - 0.25\beta^2)^{1/2}. \quad (5.18)$$

If the magnitude of the total frictional force T is fixed, the kinematic parameter v_x^0/V can be determined from equation (5.17). In Figs 3–5, boundaries of the adhesion zones and the variations of T and $\tau_{\max}$ with v_x^0/V are depicted for different values of the non-dimensional parameters β and B . In Fig. 6 we have plotted, for different values of $T' = T/(\mu c_{33}a^4/Rk)$, the variation of kinematic parameter v_x^0/V with B .

We now study the evolution of motion of the homogeneous spherical ball under the action of a normal force P and a moment M_0 (e.g. see Fig. 7) and set $\Omega = \Omega_y$, $I = I_y$, $V = V_x$

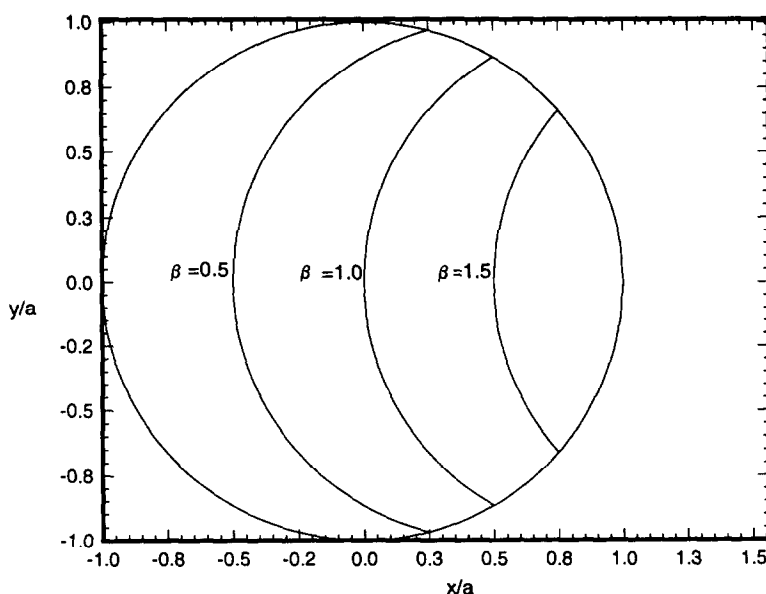


Fig. 3. Boundaries of adhesion/slip zones for different values of the non-dimensional parameter $\beta \equiv Bv_x^0/V$.

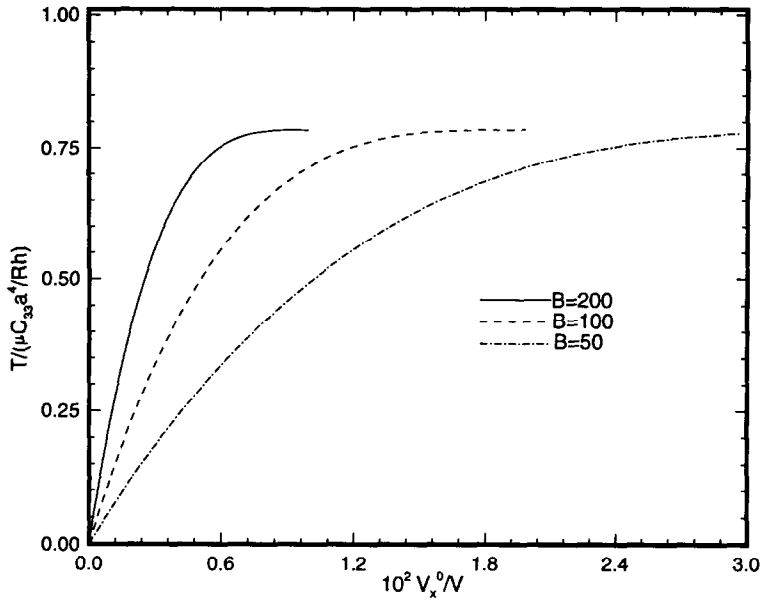


Fig. 4. Variation of the resultant frictional force with the kinematic variable v_x^0/V for different values of B .

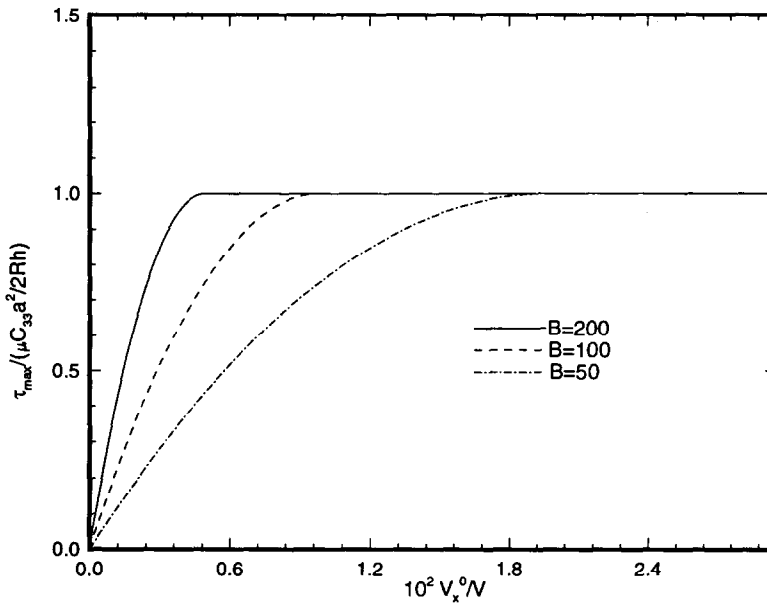


Fig. 5. Variation of the maximum value of the tangential tractions at the contact surface with the kinematic variable v_x^0/V for different values of B .

to shorten the notation. Equations (2.7) and (2.8) take the form

$$M\dot{V} = -T, \quad (5.19)$$

$$I\dot{\Omega} = TR + M_0. \quad (5.20)$$

From (5.19), (5.20) and the definition of rigid slip velocity, we obtain

$$M\dot{v}_x^0 = -3.5|T| + \frac{2.5M_0}{R}, \quad (5.21)$$

where the function T of v_x^0/V is given by (5.17), and we have used

$$\frac{MR^2}{I} = 2.5 \quad (5.22)$$

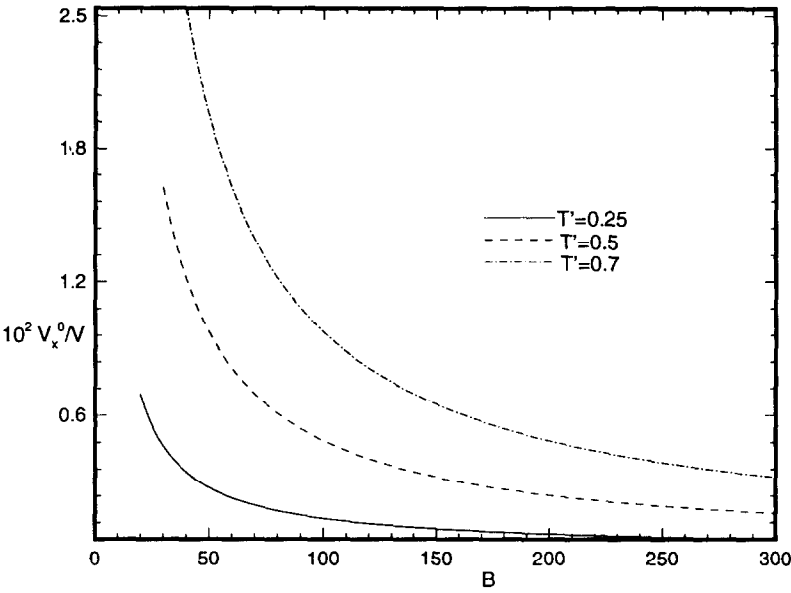


Fig. 6. Variation of the rigid slip velocity with B for different values of the resultant frictional force.

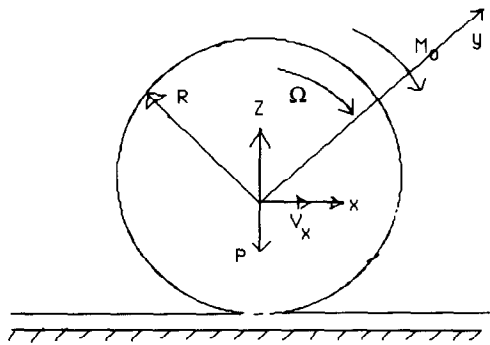


Fig. 7. Schematic sketch of the evolution problem studied.

for a homogeneous spherical ball. From (5.19) and (5.21) we conclude that

$$V(t) = V(0) + \frac{2}{7} [v_x^0(t) - v_x^0(0)] + \frac{5}{7} \frac{M_0 t}{MR}, \tag{5.23}$$

which relates the speed of the center of mass of the ball to its slip velocity. This is an example of the interaction between the contact forces and the motion of ball through v_x^0 and V . The evolution of the kinematics depends upon contact stresses through the resultant frictional force.

The non-stationary problem involving the frictional contact between a moving elastic body and a stationary one has been studied by Spector and Batra [17]. They showed that the evolution of the motion of the body can be split into two phases. During the first phase, which is of relatively short duration, surface tractions at the contact surfaces evolve rapidly, and the center of mass velocity and the angular velocity of the body can be taken to be constants. The frictional forces and the slip velocity approach limiting values which serve as initial conditions for the outer phase of the solution of the problem. The evolution of v_x^0 in the first phase obeys

$$-3.5F \left(\frac{v_x^0(t)}{V(0)} \right) + 2.5 \frac{M_0}{R} = 0, \tag{5.24}$$

where F is the resultant frictional force at the contact surface and is computed from the

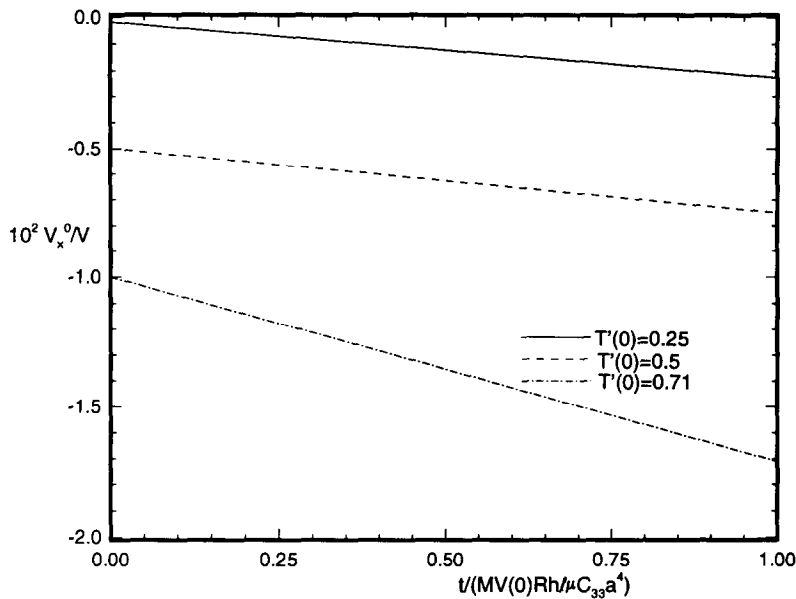


Fig. 8. Evolution of the rigid slip velocity for different values of the applied resultant tractions.

solution of the non-stationary problem. For limiting values of v_x^0 , we can replace F in (5.24) by T given by (5.17).

The processes studied here correspond to the outer phase, which means that the initial condition for (5.21) should be determined from (5.24) and hence it depends on the magnitude of the applied moment.

The evolution of the rigid slip velocity obtained by integration of (5.21) and plotted in Fig. 8 indicates that it increases almost linearly with time and its magnitude increases with the magnitude of the applied moment.

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APPENDIX

$$[\mathbf{D}] = [\mathbf{C}(\xi, \eta, 0)][\mathbf{C}^{-1}(\xi, \eta, h)] = [\mathbf{c}][\mathbf{d}(h, r)][\mathbf{c}]^{-1},$$

$$[\mathbf{c}] = \begin{bmatrix} v_1 & -v_1 & v_2 & -v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ f(v_1) & f(v_1) & f(v_2) & f(v_2) & 0 & 0 \\ g(v_1) & g(v_1) & g(v_2) & g(v_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -h(v_1) & h(v_1) & -h(v_2) & h(v_2) & 0 & 0 \end{bmatrix},$$

$$[\mathbf{d}(h, r)] = \text{diag}(e^{-v_1 rh}, e^{v_1 rh}, e^{-v_2 rh}, e^{v_2 rh}, e^{-v_3 rh}, e^{v_3 rh}).$$

$$[\mathbf{c}]^{-1} = \frac{1}{2} \begin{bmatrix} \frac{h(v_2)}{d_1} & 0 & \frac{g(v_2)}{d_2} & -\frac{f(v_2)}{d_2} & 0 & \frac{v_2}{d_1} \\ -\frac{h(v_2)}{d_1} & 0 & \frac{g(v_2)}{d_2} & -\frac{f(v_2)}{d_2} & 0 & -\frac{v_2}{d_1} \\ -\frac{h(v_1)}{d_1} & 0 & -\frac{g(v_1)}{d_2} & \frac{f(v_1)}{d_2} & 0 & -\frac{v_1}{d_1} \\ \frac{h(v_1)}{d_1} & 0 & -\frac{g(v_1)}{d_2} & \frac{f(v_1)}{d_2} & 0 & \frac{v_1}{d_1} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix},$$

where

$$f(v) = \frac{c_{11} - c_{44}v^2}{c_{13} + c_{44}}, \quad h(v) = v \left[1 - \frac{c_{33}}{c_{13}} f(v) \right], \quad g(v) = \frac{c_{11} + c_{13}v^2}{c_{13} + c_{44}},$$

$$d_1 = v_1 h(v_2) - v_2 h(v_1), \quad d_2 = f(v_1)g(v_2) - f(v_2)g(v_1).$$