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MIXED VARIATIONAL PRINCIPLES IN NON-LINEAR ELECTROELASTICITY

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Abstract—We discuss several mixed variational principles which generalize existing variational principles in the non-linear theory of electroelasticity.

1. INTRODUCTION

Piezoelectric ceramics used in smart structures are often subjected to large electric fields. Tiersten [1] has derived the fully non-linear theory of electroelasticity, has specialized it to the case of large electric fields but infinitesimal mechanical deformations [2], and has derived the corresponding non-linear two-dimensional plate equations [3]. This paper is concerned with various mixed variational principles for the non-linear theory of electroelasticity. Mixed variational principles are important in theory and many applications in elasticity [4], for example, in the solution of problems by the finite element method, the mixed variational principles help determine accurately the stresses, displacements and the electric field. A few mixed variational principles for the linear electromagnetics have recently been proposed by Felippa and Schular [5]. Variational principles for the linear theory of electroelasticity, or the theory of piezoelectricity, are of interest because of the study of smart structures [6], and have been proposed [7, 8]. In this paper, for the fully non-linear theory, existing variational principles involving elastic stress tensor and electric field or polarization vector are generalized to obtain mixed variational principles for all field variables and constraints have been accounted for by the method of Lagrange multipliers. Variational principles in terms of the total stress tensor and electric displacement vector are also derived; these are more consistent in form with the mixed variational principles of the linear theory.

2. GENERALIZATION OF TOUPIN'S VARIATIONAL FORMULATION

In the reference configuration, let the coordinates of a material particle with respect to a rectangular Cartesian coordinate system be X_k , the spatial region occupied by the elastic dielectric be V , the boundary surface of V be S , and the unit exterior normal to S be N_k . In the current configuration, let the Cartesian coordinates of the material particle be x_k , and the spatial region occupied by the elastic dielectric be v . The deformation of the material is described by the function $x(X)$ which also depends on time t for dynamic problems. Throughout this paper, a repeated index implies summation over the range of the index, and a comma followed by an index k or K stands for partial differentiation with respect to

x_k or X_K . Toupin's non-linear elastic dielectric material [9] is characterized by the following functional [10]:

$$\Gamma(\mathbf{x}, \mathbf{P}, \phi) = \int_V [-\rho \Sigma(x_{k,L}, \pi_k) - \phi_{,k} P_k + \frac{1}{2} \varepsilon_0 \phi_{,k} \phi_{,k} + \rho f_k x_k] dv \quad (1)$$

where ϕ is the electric potential related to the electric field E_k by $E_k = -\phi_{,k}$, ρ is the mass density in the current configuration, ε_0 is the permittivity of free space, P_k is the electric polarization vector, $\pi_k = P_k/\rho$ is the polarization per unit mass, f_k is the prescribed body force per unit mass, and Σ is an energy density function. The term $\varepsilon_0 \phi_{,k} \phi_{,k}/2$ represents the part of the energy contained exclusively in the electric field which is independent of material behavior, and the remaining terms represent the part of the energy due to the polarization and deformation of the dielectric body. With [11]

$$\begin{aligned} \mathcal{T} &= \det(x_{i,K}), & dv &= \mathcal{T} dV, & \rho_0 &= \rho \mathcal{T} \\ \mathcal{E}_L &= E_k x_{k,L} = -\phi_{,L} \\ \Pi_K &= \mathcal{T} X_{K,l} P_l = \rho_0 X_{K,l} \pi_l \\ E_{KL} &= \frac{1}{2} (x_{k,K} x_{k,L} - \delta_{KL}), \end{aligned} \quad (2)$$

equation (1) can be written in the referential configuration as

$$\Gamma(\mathbf{x}, \Pi, \phi) = \int_V [-\rho_0 \Sigma(E_{KL}, \Pi_K) - \phi_{,K} \Pi_K + \frac{1}{2} \varepsilon_0 \mathcal{T} X_{L,k} X_{M,k} \phi_{,L} \phi_{,M} + \rho_0 f_k x_k] dV, \quad (3)$$

where we have made use of the fact that $\Sigma(x_{k,L}, \pi_k)$ must have the form $\Sigma(E_{KL}, \Pi_K)$ for it to be an objective function of its arguments [12]. In (2), \mathcal{E}_K and Π_K are the electric field and the polarization vector in the material form, E_{KL} the Lagrange–Green strain tensor, ρ_0 the mass density in the reference configuration, and δ_{KL} the Kronecker delta. In (3), $X_{K,k}$ is understood to be determined by

$$x_{i,K} X_{K,k} = \delta_{ik}. \quad (4)$$

With

$$\mathcal{L}(\mathbf{x}, \phi) = \frac{1}{2} \varepsilon_0 \mathcal{T} X_{L,k} X_{M,k} \phi_{,L} \phi_{,M} \quad (5)$$

equation (3) becomes

$$\Gamma(\mathbf{x}, \Pi, \phi) = \int_V [-\rho_0 \Sigma(E_{KL}, \Pi_K) - \phi_{,K} \Pi_K + \mathcal{L}(\mathbf{x}, \phi) + \rho_0 f_k x_k] dV. \quad (6)$$

Using (4) and the relations given in [13], $\delta\Gamma = 0$ gives

$$\begin{aligned} [T_{KL} x_{k,L} + \mathcal{T} X_{K,l} \varepsilon_0 (\phi_{,M} X_{M,k} \phi_{,N} X_{N,l} - \frac{1}{2} \phi_{,M} X_{M,m} \phi_{,N} X_{N,m} \delta_{kl})]_{,K} + \rho_0 f_k &= 0 \quad \text{in } V \\ [\Pi_K + \mathcal{T} X_{K,k} \varepsilon_0 (-\phi_{,L} X_{L,k})]_{,K} &= 0 \quad \text{in } V \\ \mathcal{E}_K - \rho_0 \frac{\partial \Sigma}{\partial \Pi_K} &= 0 \quad \text{in } V \end{aligned} \quad (7)$$

where

$$T_{KL} = \rho_0 \frac{\partial \Sigma}{\partial E_{KL}} \quad (8)$$

is the elastic stress tensor, and we have used (2)₂. Equations (7)_{1,2} assume the following more familiar form when written in the spatial description

$$\begin{aligned} [\mathcal{T}^{-1} x_{i,M} T_{ML} x_{k,L} + \varepsilon_0 (E_k E_l - \frac{1}{2} E_m E_m \delta_{kl})]_{,l} + \rho f_k &= 0 \\ (P_k + \varepsilon_0 E_k)_{,k} &= 0. \end{aligned} \quad (9)$$

The boundary terms are discussed below.

Let the boundary surface S in the reference configuration be partitioned as

$$\begin{aligned} S_x \cup S_T &= S_\phi \cup S_D = S \\ S_x \cap S_T &= S_\phi \cap S_D = \emptyset \end{aligned} \quad (10)$$

where S_x is the part of the boundary on which the final position of a material particle is prescribed, and on S_T the traction vector, on S_ϕ the electric potential, on S_D the surface free charge density are prescribed as

$$\begin{aligned} x_k &= \bar{x}_k \quad \text{on } S_x \\ N_K [T_{KL} x_{k,L} + \mathcal{F} X_{K,l} \varepsilon_0 (\phi_{,M} X_{M,k} \phi_{,N} X_{N,l} - \frac{1}{2} \phi_{,M} X_{M,m} \phi_{,N} X_{N,m} \delta_{kl})] &= \bar{T}_k \quad \text{on } S_T \\ \phi &= \bar{\phi} \quad \text{on } S_\phi \\ N_K [\Pi_K + \mathcal{F} X_{K,k} \varepsilon_0 (-\phi_{,L} X_{L,k})] &= \bar{D} \quad \text{on } S_D \end{aligned} \quad (11)$$

where \bar{T}_k and \bar{D} are measured with respect to the surface area in the reference configuration. The surface free charge density \bar{D} is usually zero.

To obtain mixed variational formulations, we write the functional (6) in the following constrained form:

$$\begin{aligned} \Gamma_1(x_k, E_{KL}, \phi, \Pi_K) &= \int_V [-\rho_0 \Sigma(E_{KL}, \Pi_K) - \phi_{,K} \Pi_K + \mathcal{L}(x, \phi) + \rho_0 f_k x_k] dV \\ &+ \int_{S_T} \bar{T}_k x_k dS + \int_{S_D} \bar{D} \phi dS \end{aligned} \quad (12)$$

with constraints

$$\begin{aligned} E_{KL} &= \frac{1}{2}(x_{k,K} x_{k,L} - \delta_{KL}) \quad \text{in } V \\ x_k &= \bar{x}_k \quad \text{on } S_x \\ \phi &= \bar{\phi} \quad \text{on } S_\phi \end{aligned} \quad (13)$$

where we have added boundary terms involving surface tractions and charge density for a complete treatment of the problem. Then, by using the method of Lagrange multipliers, we obtain the following functional:

$$\begin{aligned} \Gamma_2(x_k, E_{KL}, T_{KL}, \phi, \Pi_K) &= \int_V \{ -\rho_0 \Sigma(E_{KL}, \Pi_K) - \phi_{,K} \Pi_K + T_{KL} [E_{KL} - \frac{1}{2}(x_{k,K} x_{k,L} - \delta_{KL})] \\ &+ \mathcal{L}(x, \phi) + \rho_0 f_k x_k \} dV \\ &+ \int_{S_x} (x_k - \bar{x}_k) N_K [T_{KL} x_{k,L} + \mathcal{F} X_{K,l} \varepsilon_0 (\phi_{,M} X_{M,k} \phi_{,N} X_{N,l} \\ &- \frac{1}{2} \phi_{,M} X_{M,m} \phi_{,N} X_{N,m} \delta_{kl})] dS \\ &+ \int_{S_\phi} (\phi - \bar{\phi}) N_K [\Pi_K + \mathcal{F} X_{K,k} \varepsilon_0 (-\phi_{,L} X_{L,k})] dS \\ &+ \int_{S_T} \bar{T}_k x_k dS + \int_{S_D} \bar{D} \phi dS \end{aligned} \quad (14)$$

in which variables are not subjected to any constraints. The stationary conditions of Γ_2 are equations (7), (8) and

$$\begin{aligned} E_{KL} - \frac{1}{2}(x_{k,K} x_{k,L} - \delta_{KL}) &= 0 \quad \text{in } V \\ x_k &= \bar{x}_k \quad \text{on } S_x \\ N_K [T_{KL} x_{k,L} + \mathcal{F} X_{K,l} \varepsilon_0 (\phi_{,M} X_{M,k} \phi_{,N} X_{N,l} - \frac{1}{2} \phi_{,M} X_{M,m} \phi_{,N} X_{N,m} \delta_{kl})] &= \bar{T}_k \quad \text{on } S_T \\ \phi &= \bar{\phi} \quad \text{on } S_\phi \\ N_K [\Pi_K + \mathcal{F} X_{K,k} \varepsilon_0 (-\phi_{,L} X_{L,k})] &= \bar{D} \quad \text{on } S_D. \end{aligned} \quad (15)$$

Equations (7), (8) and (15) are the complete set of equations and boundary conditions for non-linear electroelasticity.

The functional Γ_2 can be used to generate other functionals through Legendre transforms. For example, with the introduction of

$$\rho_0 W(E_{KL}, \mathcal{E}_K) = \rho_0 \Sigma(E_{KL}, \Pi_K) - \mathcal{E}_K \Pi_K \quad (16)$$

Γ_2 is changed to a functional of x_k , E_{KL} , T_{KL} , ϕ , \mathcal{E}_K and Π_K , with the constraint $\mathcal{E}_K = -\phi_{,K}$. We account for this constraint by using Lagrange multipliers and obtain equations (7)_{1,2}, (15) and

$$\begin{aligned} \Pi_K + \rho_0 \frac{\partial W}{\partial \mathcal{E}_K} &= 0 \quad \text{in } V \\ T_{KL} - \rho_0 \frac{\partial W}{\partial E_{KL}} &= 0 \quad \text{in } V \\ \mathcal{E}_K + \phi_{,K} &= 0 \quad \text{in } V \end{aligned} \quad (17)$$

as the stationary conditions which are equivalent to (7), (8) and (15). This formulation is very general. It can be used to derive other variational principles. For example, if we use (17)₃ and (15)₁ to express E_{KL} and \mathcal{E}_K in terms of x_k and ϕ , and require x_k and ϕ to satisfy (15)_{2,4}, we obtain the following functional:

$$\begin{aligned} \Gamma_4(x_k, \phi) &= \int_V [-\rho_0 W(E_{KL}, \mathcal{E}_K) + \mathcal{L}(x, \phi) + \rho_0 f_k x_k] dV \\ &\quad + \int_{S_T} \bar{T}_k x_k dS + \int_{S_D} \bar{D} \phi dS \end{aligned} \quad (18)$$

which is the functional used in [14, 15] when boundary and body force terms are dropped.

3. FORMULATIONS USING THE TOTAL ENERGY DENSITY

We observe that the energy density \mathcal{L} defined by (5) due to the electric field alone can be written as a function of E_{KL} and \mathcal{E}_K as follows:

$$\begin{aligned} \mathcal{L}(E_{KL}, \mathcal{E}_K) &= \frac{1}{2} \varepsilon_0 \mathcal{T} X_{L,k} X_{M,k} \phi_{,L} \phi_{,M} \\ &= \frac{1}{2} \varepsilon_0 (\mathcal{T}_C)^{1/2} C_{LM}^{-1} \mathcal{E}_L \mathcal{E}_M \end{aligned} \quad (19)$$

where

$$\begin{aligned} C_{KL} &= x_{k,K} x_{k,L} = 2E_{KL} + \delta_{KL} \\ \mathcal{T}_C &= \det(C_{KL}) \end{aligned} \quad (20)$$

and also

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial E_{KL}} &= -\mathcal{T} X_{K,k} X_{L,l} \varepsilon_0 (E_k E_l - \frac{1}{2} E_m E_m \delta_{kl}) \\ \frac{\partial \mathcal{L}}{\partial \mathcal{E}_K} &= \mathcal{T} X_{K,k} \varepsilon_0 E_k. \end{aligned} \quad (21)$$

Hence if we introduce a total energy density $W'(E_{KL}, \mathcal{E}_K)$ as

$$\rho_0 W'(E_{KL}, \mathcal{E}_K) = \rho_0 W(E_{KL}, \mathcal{E}_K) - \mathcal{L}(E_{KL}, \mathcal{E}_K) \quad (22)$$

we have

$$\begin{aligned} -\rho_0 \frac{\partial W'}{\partial \mathcal{E}_K} &= \Pi_K + \mathcal{T} X_{K,k} \varepsilon_0 E_k = \mathcal{T} X_{K,k} (P_k + \varepsilon_0 E_k) = \mathcal{T} X_{K,k} D_k = \mathcal{D}_K \\ \rho_0 \frac{\partial W'}{\partial E_{KL}} &= T_{KL} + \mathcal{T} X_{K,k} X_{L,l} \varepsilon_0 (E_k E_l - \frac{1}{2} E_m E_m \delta_{kl}) = T'_{KL} \end{aligned} \quad (23)$$

where D_k is the electric displacement vector, \mathcal{D}_K its material form, and T'_{KL} is the total stress tensor. In terms of T'_{KL} and \mathcal{D}_K , (7)_{1,2} can be written as

$$\begin{aligned} (T'_{KL} x_{k,K})_{,K} + \rho_0 f_k &= 0 \\ \mathcal{D}_{K,K} &= 0. \end{aligned} \quad (24)$$

While having clear physical interpretations, the separation of the energy into the part \mathcal{L} due to the electric field only and the part due to the dielectric body makes the mathematical manipulations complicated. In linear theory, the electric displacement vector D_k is used more often than the polarization vector P_k . Various variational principles can also be formulated in terms of the total energy density, total stress tensor and the electric displacement vector. With the total energy density, we write (18) as

$$\begin{aligned} \Gamma_5(x_k, E_{KL}, \phi, \mathcal{E}_K) = & \int_V [-\rho_0 W'(E_{KL}, \mathcal{E}_K) + \rho_0 f_k x_k] dV \\ & + \int_{S_T} \bar{T}_k x_k dS + \int_{S_D} \bar{D} \phi dS \end{aligned} \quad (25)$$

with constraints (13) and $\mathcal{E}_K = -\phi_{,K}$ in V .

Using Lagrange multipliers and the standard procedures for constrained minimization problems, we obtain as stationary conditions equations (23), (24), (15) and

$$\mathcal{E}_K + \phi_{,K} = 0 \quad \text{in } V \quad (26)$$

which become (7)_{1,2}, (15) and (17) once we specify that W' is given by (22). If a Σ' is introduced through Legendre transform

$$\rho_0 \Sigma'(T'_{KL}, \mathcal{D}_K) = \rho_0 W'(E_{KL}, \mathcal{E}_K) + \mathcal{E}_K \mathcal{D}_K - T'_{KL} E_{KL} \quad (27)$$

we obtain a four-field functional in terms of x_k , T'_{KL} , ϕ , and \mathcal{D}_K whose stationary conditions are (24) and

$$\begin{aligned} \phi_{,K} + \rho_0 \frac{\partial \Sigma'}{\partial \mathcal{D}_K} &= 0 \quad \text{in } V \\ \frac{1}{2} (x_{k,K} x_{k,L} - \delta_{KL}) + \rho_0 \frac{\partial \Sigma'}{\partial T'_{KL}} &= 0 \quad \text{in } V \\ x_k &= \bar{x}_k \quad \text{on } S_x \\ N_K T'_{KL} x_{k,L} &= \bar{T}_k \quad \text{on } S_T \\ \phi &= \bar{\phi} \quad \text{on } S_\phi \\ N_K \mathcal{D}_K &= \bar{D} \quad \text{on } S_D \end{aligned} \quad (28)$$

which are equivalent to (23), (24), (15) and (26). Different functionals introduced above may be considered as generalizations of those studied in linear piezoelectricity [8], or as generalizations of the ones for non-linear elasticity studied in [4].

Finally, we note that variational formulations for dynamic problems can be obtained by adding a kinetic energy term and integrating over a time interval, for example:

$$\begin{aligned} \Gamma_8(x_k, E_{KL}, T'_{KL}, \phi, \mathcal{D}_K, \mathcal{E}_K) \\ = \int_{t_0}^{t_1} dt \int_V \{ \frac{1}{2} \rho_0 \dot{x}_k \dot{x}_k - \rho_0 W'(E_{KL}, \mathcal{E}_K) - \mathcal{D}_K (\mathcal{E}_K + \phi_{,K}) \\ + T'_{KL} [E_{KL} - \frac{1}{2} (x_{k,K} x_{k,L} - \delta_{KL})] + \rho_0 f_k x_k \} dV \\ + \int_{t_0}^{t_1} dt \int_{S_x} (x_k - \bar{x}_k) N_K T'_{KL} x_{k,L} dS \\ + \int_{t_0}^{t_1} dt \int_{S_\phi} (\phi - \bar{\phi}) N_K \mathcal{D}_K dS \\ + \int_{t_0}^{t_1} dt \int_{S_T} \bar{T}_k x_k dS + \int_{t_0}^{t_1} dt \int_{S_D} \bar{D} \phi dS \end{aligned} \quad (29)$$

where a dot superimposed on a quantity indicates its material time derivative. The stationary conditions for Γ_8 are

$$(T_{KL}^i x_{k,L})_{,K} + \rho_0 f_k = \rho_0 \ddot{x}_k \quad \text{in } V, \quad t_0 < t < t_1$$

$$\mathcal{D}_{K,K} = 0 \quad \text{in } V, \quad t_0 < t < t_1$$

$$\mathcal{D}_K + \rho_0 \frac{\partial W^i}{\partial \mathcal{E}_K} = 0 \quad \text{in } V, \quad t_0 < t < t_1$$

$$T_{KL}^i - \rho_0 \frac{\partial W^i}{\partial E_{KL}} = 0 \quad \text{in } V, \quad t_0 < t < t_1$$

$$\mathcal{E}_K + \phi_{,K} = 0 \quad \text{in } V, \quad t_0 < t < t_1$$

$$E_{KL} - \frac{1}{2}(x_{k,K} x_{k,L} - \delta_{KL}) = 0 \quad \text{in } V, \quad t_0 < t < t_1$$

$$x_k = \bar{x}_k \quad \text{on } S_k, \quad t_0 < t < t_1$$

$$N_K T_{KL}^i x_{k,L} = \bar{T}_k \quad \text{on } S_T, \quad t_0 < t < t_1$$

$$\phi = \bar{\phi} \quad \text{on } S_\phi, \quad t_0 < t < t_1$$

$$N_K \mathcal{D}_K = \bar{D} \quad \text{on } S_D, \quad t_0 < t < t_1$$

when

$$\delta x_k|_{t=t_0} = \delta x_k|_{t=t_1} = 0. \quad (31)$$

4. CONCLUSIONS

We have considered various functionals and shown that the vanishing of the first variations gives the pertinent governing equations and boundary conditions. Thus the solution of a boundary-value problem or an initial-boundary value problem entails finding the stationary value of a functional. These variational principles are mixed in the sense that stresses, electric field, displacements and electric potential, etc., are considered as field variables. These mixed variational principles are often employed in the finite element solutions of problems involving cracks or other points of discontinuities.

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