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Comparison of results from four linear constitutive relations in isotropic finite elasticity

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Abstract

We use four linear constitutive relations to study finite deformations of a biaxially loaded elastic membrane, triaxially loaded cube, and simple extensional and simple shearing deformations of an elastic body. In each case, the body is assumed to be isotropic and homogeneous. It is shown that only the neoHookean relation (a linear relation between the Cauchy stress tensor and the left Cauchy–Green tensor) and the Signorini's relation (a linear relationship between the Cauchy stress tensor and the Almansi–Hamel strain tensor) predict load–deformation curves that qualitatively agree with most of the test observations. A similar conclusion holds when the body is assumed to be incompressible. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Biaxially loaded membrane; Load–deformation relations; Material instability; Simple extensional deformations; Simple shear

1. Introduction

In a previous note [1] we considered the following two linear constitutive relations to study the simple extension and simple shear deformations of a homogeneous isotropic elastic body:

$$\mathbf{S} = \lambda(\mathrm{tr}\,\mathbf{E})\mathbf{1} + 2\mu\mathbf{E}, \tag{1}$$

$$\boldsymbol{\sigma} = \frac{\lambda}{2}(\mathrm{tr}(\mathbf{B}-\mathbf{1}))\mathbf{1} + \mu(\mathbf{B}-\mathbf{1}). \tag{2}$$

Here $\mathbf{S}$ is the second Piola–Kirchhoff stress tensor, $\mathbf{E}$ the Green–St. Venant strain tensor, $\mathbf{B}$ the left Cauchy–Green tensor, $\boldsymbol{\sigma}$ the Cauchy stress, $\mathbf{1}$ the identity tensor, tr the trace operator, and λ and μ are Lamé constants which satisfy $\lambda + \frac{2}{3}\mu > 0$ and $\mu > 0$. It was shown that the axial nominal stress vs. axial stretch curve in simple extension and the shear stress vs. the shear strain curve in simple shearing deformations predicted from constitutive relation (1) do not qualitatively agree with those observed experimentally. However, the corresponding curves obtained from constitutive relation (2) are in qualitative agreement with the test observations. Here, in addition to (1) and (2) we use the following two constitutive relations to study the aforestated deformations of an elastic body as well as deformations of a biaxially loaded isotropic elastic membrane, and a triaxially loaded cube:

$$\boldsymbol{\sigma} = \lambda(\mathrm{tr}\,\mathbf{A})\mathbf{1} + 2\mu\mathbf{A}, \tag{3}$$

$$\bar{\mathbf{T}} = \lambda(\mathrm{tr}\,\mathbf{A}_\ell)\mathbf{1} + 2\mu\mathbf{A}_\ell,\ \bar{\mathbf{T}} = J\mathbf{R}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{R},\ \mathbf{A}_\ell = \ln\mathbf{U}. \tag{4}$$

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In Eqs. (3) and (4), $\mathbf{A} = (\mathbf{1} - \mathbf{B}^{-1})/2$ is the Almansi–Hamel strain tensor, $\mathbf{U}$ is the right stretch tensor in the polar decomposition $\mathbf{F} = \mathbf{RU}$ of the deformation gradient $\mathbf{F}$, $\mathbf{R}$ is the rotation tensor, and $J = \det \mathbf{F}$. Note that $\mathbf{B} = \mathbf{FF}^{\mathrm{T}}$, and $\mathbf{E} = (\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{1})/2$. The eigenvalues of the logarithmic strain tensor $\mathbf{A}_{\ell}$ equal the natural logarithm of the eigenvalues of $\mathbf{U}$, and the two tensors have the same eigenvectors. The first Piola–Kirchhoff stress tensor, $\mathbf{T}$, is related to $\boldsymbol{\sigma}$ and $\mathbf{S}$ as follows:

$$\boldsymbol{\sigma} = \mathbf{TF}^{\mathrm{T}}/J, \quad \mathbf{T} = \mathbf{FS}. \tag{5}$$

For infinitesimal deformations, each one of the constitutive relations (1)–(4) reduces to Hooke's law. Also, each one of these four relations is objective and is also invertible in the sense that a strain tensor can be expressed in terms of the corresponding stress tensor. The principal axes of stress and strain coincide, and a triaxial state of stress at a point will produce a triaxial state of strain. Constitutive relations (1) and (4) can also be expressed in terms of $\boldsymbol{\sigma}$ and $\mathbf{B}$ but such relations will be non-linear.

Constitutive relation (1) was proposed by St. Venant and Kirchhoff (e.g. see [2]), (3) is a special case of that proposed by Signorini, (2) is usually known as the neoHookean material, and (4) has been studied by Hill [3]. The deformations studied herein are homogeneous; thus the balance of linear momentum is trivially satisfied.

2. Unconstrained materials

2.1. Simple shear

In rectangular Cartesian coordinates, consider the simple shearing deformation

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{6}$$

where $\mathbf{x}$ gives the present position of the material particle that occupied place $\mathbf{X}$ in the reference configuration and k is a constant. For the four constitutive relations (1)–(4), the shear stress T_{12} is related to the shear strain k as follows:

$$\begin{aligned}
T_{12}^{(1)} &= \mu k + \left(\frac{\lambda}{2} + \mu\right)k^3,\\
T_{12}^{(2)} &= \mu k,\\
T_{12}^{(3)} &= \mu k,\\
T_{12}^{(4)} &= \frac{\mu}{\sqrt{1 + k^2/4}}\ln\left(1 + \frac{k^2}{2} + k\sqrt{1 + k^2/4}\right).
\end{aligned} \tag{7}$$

Here $T_{12}^{(1)}$ denotes the value of T_{12} for the constitutive relation (1). For $\lambda/\mu = 20$, the normalized shear stress, T_{12}/μ, vs. the shear strain, k, is plotted in Fig. 1 for the four constitutive relations. Whereas for constitutive relations (2)–(4), the shear stress depends only upon the Lamé constant μ, for constitutive relation (1), it also depends upon λ. For rubber-like materials, $\lambda > \mu$. Note that the shear stress vs. the shear strain curve is concave upwards for constitutive relation (1), it is concave downwards for constitutive relation (4) and is linear for the other two constitutive relations. The shear stress monotonically increases with k for constitutive relations (1)–(3) but attains a maximum value of 1.3255μ at $k = 3.0178$ for constitutive relation (4). Adopting the Considère criterion [4], i.e., the material becomes unstable when the applied load is maximum, the material described by (4) will

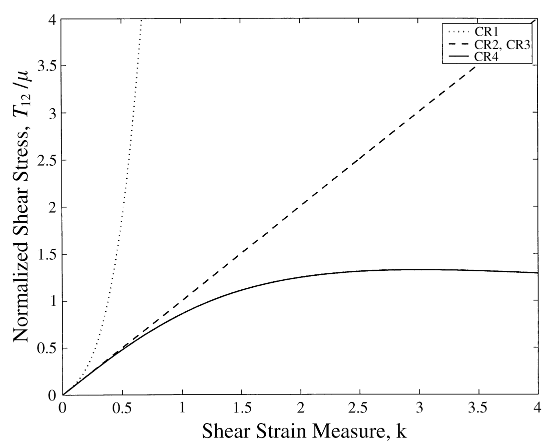

Fig. 1. The normalized shear stress, T_{12}/μ, vs. the shear strain, k, for simple shearing deformations.

become unstable at $k = 3.0178$. The normal stress on planes $X_2 = \text{const.}$ is given by

$$T_{22}^{(1)} = \left(\frac{\lambda}{2} + \mu\right)k^2,$$

$$T_{22}^{(2)} = \frac{\lambda}{2}k^2,$$

$$T_{22}^{(3)} = -\left(\frac{\lambda}{2} + \mu\right)k^2,$$

$$T_{22}^{(4)} = -\frac{\mu k}{\sqrt{4+k^2}}\ln\left(1 + \frac{k^2}{2} + k\sqrt{1+k^2/4}\right). \tag{8}$$

For $\mu > 0$ and $\lambda > 0$, constitutive relations (1) and (2) require that a tensile normal stress be applied to planes $X_2 = \text{const.}$ in order to produce simple shear. However, constitutive relations (3) and (4) require that this normal stress be compressive. Thus, in the absence of these normal stresses, the Poynting effect (e.g. see [2]) predicted by constitutive relations (1) and (2) is opposite to that given by constitutive relations (3) and (4). For $\lambda/\mu = 20$, the magnitude of the Poynting effect predicted by constitutive relation (4) is much smaller than that for constitutive relations (1)–(3). For constitutive relations (1)–(3), the magnitude of the normal stress and hence of the Poynting effect is proportional to k^2; for constitutive relation (4) this holds for $k \ll 1$. The normal stresses on planes $X_1 = \text{const.}$ and $x_1 = \text{const.}$ are also different for each constitutive relation.

2.2. Simple extension

Following the usual approach (see e.g. [1]) we obtain the following relations (9) between the axial load, P, per unit undeformed area and the corresponding axial stretch, α, in a prismatic isotropic and homogeneous elastic body deformed in simple extension:

$$P^{(1)} = \mu\frac{3\lambda + 2\mu}{2(\lambda+\mu)}\alpha(\alpha^2 - 1),$$

$$P^{(2)} = \mu\left(\frac{3\lambda + 2\mu}{2(\lambda+\mu)}\right)^2(\alpha^2 - 1)\left(1 - \frac{\lambda}{3\lambda + 2\mu}\alpha^2\right),$$

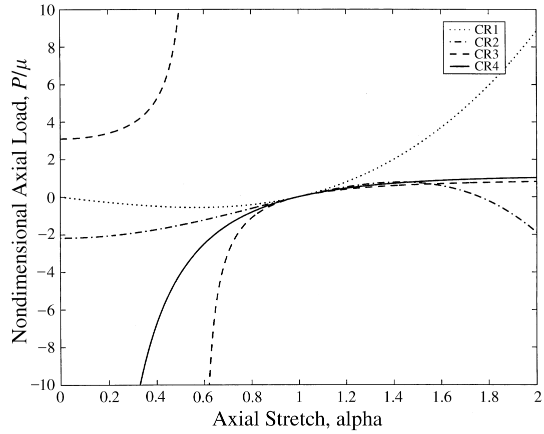

Fig. 2. The normalized axial load, P/μ, vs. the axial stretch, α, for simple extensional deformations.

$$P^{(3)} = \mu\left(1 + \frac{2(\lambda+\mu)}{\lambda - (3\lambda + 2\mu)\alpha^2}\right),$$

$$P^{(4)} = \mu\frac{3\lambda + 2\mu}{\lambda + \mu}\,\frac{1}{\alpha}\ln\alpha. \tag{9}$$

These correspond respectively to the constitutive relations (1)–(4); the load deformation curves are exhibited in Fig. 2 for a material with $\lambda/\mu = 20$. According to the Considère condition [4], an instability occurs when the axial load per unit undeformed cross-sectional area reaches an extreme value. We find that an instability initiates when $\alpha = 0.577$, $\sqrt{(2\lambda + \mu)/\lambda}$ and 2.718 for constitutive relations (1), (2) and (4), respectively. For the constitutive relation (3), $\mathrm{d}P/\mathrm{d}\alpha > 0$, and the load is a monotonically increasing function of the stretch. The load increases from $\mu(3\lambda + 2\mu)/\lambda$ at $\alpha \to 0$ to ∞ for $\alpha = (\lambda/(3\lambda + 2\mu))^{1/2} \equiv \alpha^*$. This is unphysical because an axial tensile load is predicted by the constitutive relation for extremely large compressive deformations. However, for $\alpha > \alpha^*$, the load monotonically increases from $-\infty$ to μ with an increase in the value of the axial stretch. Thus, the load is discontinuous at $\alpha = \alpha^*$ and the constitutive relation predicts unphysical behavior for stretches close to α^*. For constitutive relations

(1), (2) and (4), the axial load, P_{ins}, at the initiation of an instability is respectively given by

$$P_{\text{ins}}^{(1)} = -\frac{\mu}{\sqrt{3}}\frac{\lambda + 2\mu/3}{\lambda + \mu},$$

$$P_{\text{ins}}^{(2)} = \frac{\mu}{4\lambda}(3\lambda + 2\mu),$$

$$P_{\text{ins}}^{(4)} = \frac{\mu}{2.718}\frac{3\lambda + 2\mu}{\lambda + \mu}. \tag{10}$$

Hence, the magnitude of the axial nominal stress at instability is of the order of the magnitude of the Lamé constants.

The Poisson function (e.g. see [9]), $\nu(\alpha)$ ($= (1-\beta)/(\alpha - 1)$, where β is the stretch in a lateral direction), for the four constitutive relations (1)–(4) has the expressions

$$\nu^{(1)}(\alpha) = \left(1 - \left(\frac{3\lambda + 2\mu}{2(\lambda+\mu)} - \frac{\lambda\alpha^2}{2(\lambda+\mu)}\right)^{1/2}\right)\Big/(\alpha - 1),$$

$$\nu^{(2)}(\alpha) = \left(1 - \left(\frac{3\lambda + 2\mu}{2(\lambda+\mu)} - \frac{\lambda\alpha^2}{2(\lambda+\mu)}\right)^{1/2}\right)\Big/(\alpha - 1),$$

$$\nu^{(3)}(\alpha) = (1 - ((\lambda+\mu)/((3-\alpha^{-2})\lambda/2 + \mu))^{1/2})/$$
$$(\alpha - 1),$$

$$\nu^{(4)}(\alpha) = (1 - \alpha^{-\lambda/2(\lambda+\mu)})/(\alpha - 1),$$

and its variation with α is depicted in Fig. 3. Note that the Poisson ratio defined as

$$\nu_0 = \lim_{\alpha \to 1} \nu(\alpha) = \frac{\lambda}{2(\lambda + \mu)},$$

has the same value for each one of the four constitutive relations. However, the Poisson function for constitutive relations (1) and (2) is the same but it differs noticeably from those for constitutive relations (3) and (4). When $\nu(\alpha) > 1$, the contraction per unit length of line elements in the lateral direction will be greater than the elongation per unit length in the axial direction. For a homogeneous isotropic elastic body, the usual expectation is that $0 < \nu(\alpha) < 1$. Whereas constitutive relations (1) and (2) violate these inequalities on $\nu(\alpha)$ for simple extensional deformations with $\alpha \geqslant 1.7097$, constitutive relations (3) and (4) do so for $\alpha < 0.7587$ and 0.3488, respectively.

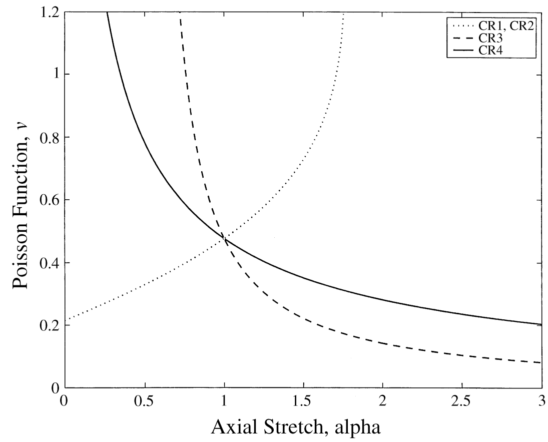

Fig. 3. The Poisson function, ν, vs. the axial stretch, α, during simple extensional deformations of a prismatic body.

2.3. *Biaxial loading of a membrane*

2.3.1. *Equal dead loads on the edges*

In this subsection, we follow Müller [5] who studied the stability of a biaxially loaded elastic membrane made of a Mooney–Rivlin material. Assume that dead loads F_1 and F_2 per unit undeformed length are applied, respectively, in the x_1 and x_2 directions to the edges of an isotropic and homogeneous rectangular membrane whose thickness in the unstressed reference configuration is uniform and equals t_0. There are no loads applied to the top and bottom surfaces of the membrane. Thus

$$F_1 = t_0 T_{11}, \tag{11a}$$

$$F_2 = t_0 T_{22}, \tag{11b}$$

$$F_3 = 0 = T_{33}. \tag{11c}$$

Here and below we use rectangular Cartesian coordinates with x_3-axis in the thickness direction. Let the homogeneous stretches produced in the membrane along the x_1, x_2 and x_3 axes be α, β and γ, respectively. Evaluating the stretch γ from the boundary condition (11c), requiring that $\infty > \gamma > 0$, and setting $F_1 = F_2$, we arrive at the following relations between α and β for the constitutive relations (1)–(4).

$$\text{CR1:}\quad (\alpha-\beta)[\delta(\alpha^2+\beta^2-2)+\alpha^2+\beta^2 + \alpha\beta - 1] = 0, \tag{12a}$$

$$\alpha^2+\beta^2-3 < (1-\delta)/\delta, \tag{12b}$$

$$\text{CR2:}\quad (\alpha-\beta)[\delta(\alpha^2+\beta^2-2)-(1+\alpha\beta)] = 0, \tag{12c}$$

$$\alpha^2+\beta^2-3 < (1-\delta)/\delta, \tag{12d}$$

$$\text{CR3:}\quad (\alpha-\beta)[\delta(\alpha^2+\beta^2-2\alpha^2\beta^2)+\alpha^2+\beta^2 + \alpha\beta - \alpha^2\beta^2] = 0, \tag{12e}$$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}-3 < (1-\delta)/\delta, \tag{12f}$$

$$\text{CR4:}\quad (\alpha-\beta)\delta\ln(\alpha\beta)+\ln(\beta^\alpha/\alpha^\beta) = 0. \tag{12g}$$

The prefix CR1 implies that the relation following it between α and β is for the constitutive relation 1. In (12a)–(12g) $\delta = \lambda/(\lambda+2\mu)$. The constraints (12b), (12d) and (12f) on the values of α and β are imposed by the requirement that $\infty > \gamma > 0$. The constitutive relation (4) does not restrict the admissible values of α and β. Note that in each case, $\alpha = \beta$ is a solution implying that equal axial edge loads per unit undeformed length produce identical stretches in the loading directions. However, there may be other solutions for which $\alpha \neq \beta$ and which depend upon the value of δ. Regarding β as a positive-valued function of α, we conclude from (12a), (12c) and (12e) that unequal axial stretches with equal biaxial loading are possible provided that

$$\text{CR1:}\quad (1+\delta)\alpha^2-(1+2\delta) < 0,$$

$$\text{CR2:}\quad (1-4\delta^2)\alpha^2+4\delta(1+2\delta) > 0,$$

$$\text{CR3:}\quad (1+2\delta)\alpha^2-(1+\delta) > 0, \tag{13}$$

respectively, for constitutive relations (1)–(3). The corresponding relations between α and β are

CR1:

$$\beta = \frac{-\alpha+\sqrt{\alpha^2-4(1+\delta)[(1+\delta)\alpha^2-(1+2\delta)]}}{2(1+\delta)}, \tag{14a}$$

CR2:

$$\beta = \frac{\alpha \pm \sqrt{\alpha^2-4\delta(\delta\alpha^2-2\delta-1)}}{2\delta}, \tag{14b}$$

CR3:

$$\beta^{-1} = \frac{-\alpha^{-1}+\sqrt{\alpha^{-2}+4(1+\delta)[(1+2\delta)-(1+\delta)\alpha^{-2}]}}{2(1+\delta)}. \tag{14c}$$

Note that α and β must also satisfy inequalities (12b) (12d) and (12f), respectively, for constitutive relations (1)–(3). The solution (14b) with the minus sign before the square root is valid only when $1+2\delta-\delta\alpha^2 < 0$. Multiple roots of Eq. (12g), if they exist, can be found numerically. For $\lambda = 20\mu$, Eq. (12a) also has unequal roots for $0 < \alpha < 1$, Eq. (12e) for $\alpha \geqslant 0.823$, and the other two Eqs. (12c) and (12g) do not have unequal roots. In order to ascertain which one of the multiple solutions is realized experimentally, we need to investigate the stability of these solutions. For stretches prescribed on the edges, the solution that gives lower value of the stored energy will be stable. However, if dead loads are prescribed on the edges, then the configuration with the lower value of the potential energy $W - F_1\alpha - F_2\beta$ will be stable; e.g. see [5]. Here $W = W(\alpha,\beta)$ is the stored energy function for the material of the membrane. Except for constitutive relation (1), the stored energy function W for the other three constitutive relations is unknown; hence the stability of solutions has not been studied. Treloar's experimental data [6, Table 1] indicates asymmetric equilibrium stretches in a rectangular rubber sheet subjected to in-plane equal biaxial loads applied perpendicular to the sheet edges.

2.3.2. Equal stretches on the edges

Consider the case of a hard loading device which applies equal stretches on all four edges of a homogeneous and isotropic rectangular membrane. Assume that the state of deformation in the membrane is such that equal stretches are produced. Then the load deformation relations for the four constitutive relations (1)–(4) are

$$F^{(1)}/(\mu t_0) = (1 + 2\delta)\alpha(\alpha^2 - 1), \tag{15a}$$

$$F^{(2)}/(\mu t_0) = (1 + 2\delta)\alpha(\alpha^2 - 1)(1 + 2\delta(1 - \alpha^2))^{1/2}, \tag{15b}$$

$$F^{(3)}/(\mu t_0) = (1 + 2\delta)\left(\alpha - \frac{1}{\alpha}\right)((1 + 2\delta) - 2\delta/\alpha^2)^{-1/2}, \tag{15c}$$

$$F^{(4)}/(\mu t_0) = 2(1 + 2\delta)(\ln\alpha)/\alpha, \tag{15d}$$

where F is the edge load per unit undeformed length. Fig. 4 evinces these load–deformation relations for $\delta = \frac{10}{11}$. The right-hand side of Eq. (15b) is real-valued only for $\alpha < 1.245$. Like the load–elongation curves for simple extension which essentially coincided for values of α close to 1.0, the load–elongation curves for the biaxially loaded membrane overlap for values of α near 1.0. The qualitative nature of the load–elongation curves for constitutive relations (1), (3) and (4) for the biaxially loaded membrane is similar to the corresponding curves for the axially loaded prismatic body. The stretch corresponding to an extreme load that also satisfies inequalities (12b), (12d) and (12f) with $\beta = \alpha$ is given by

CR1: $\alpha = 0.577$,

CR2: $8\alpha^4 - (10 + 3/\delta)\alpha^2 + (2 + 1/\delta) = 0$,

$\alpha^2 < (1 + 2\delta)/2\delta$,

CR3: no value of α,

CR4: $\alpha = e = 2.718$. (16)

Thus an instability occurs at universal stretch values of 0.577 and 2.718 in a very thin plate made of materials described by constitutive relations (1) and (4) and stretched by equal amounts in the two lateral directions. Whereas the membrane made of material (1) will become unstable in compression, that made of material (4) will experience an instability during extensional deformations. For constitutive relations (1) and (4), the load, F_{ins}, at the instant of instability equals, respectively,

$$F^{(1)}_{\text{ins}}/(\mu t_0) = -0.385(1 + 2\delta),$$

$$F^{(4)}_{\text{ins}}/(\mu t_0) = 0.736(1 + 2\delta). \tag{17}$$

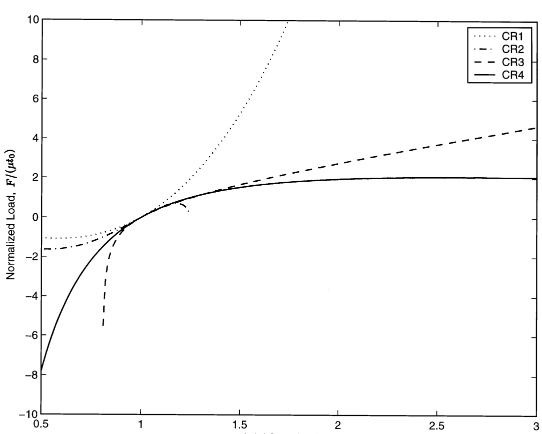

Fig. 4. The normalized axial load per unit thickness, $F/\mu t_0$, vs. the axial stretch, α, during the biaxial stretching of a membrane.

2.3.3. Different dead loads on the edges

We now investigate the case when loads (11a) and (11b) are unequal, i.e., $F_1 = \phi F_2$ with $\phi < 1$. Inequalities (12b), (12d) and (12f) still apply for constitutive relations (1)–(3), respectively. The relations between β and α for the four constitutive relations are

$$\text{CR1:}\quad \beta^3 - \frac{\alpha\delta}{\phi(1+\delta)}\beta^2 + \beta\left(\frac{\alpha^2\delta - 1 - 2\delta}{1+\delta}\right) + \frac{\alpha}{\phi}\left(\frac{1+2\delta}{1+\delta} - \alpha^2\right) = 0, \tag{18a}$$

$$\text{CR2:}\quad \beta^3 - \frac{\alpha\phi(1+\delta)}{\delta}\beta^2 + \beta\left(\frac{1+\delta}{\delta}\alpha^2 - \frac{1+2\delta}{\delta}\right) + \alpha\phi\left(\frac{1+2\delta}{\delta} - \alpha^2\right) = 0, \tag{18b}$$

$$\text{CR3:}\quad \beta^3(\alpha^2(1+2\delta) - (1+\delta)) - \beta^2(\alpha^2(1+2\delta) - \delta)\phi\alpha - \beta\delta\alpha^2 + \phi\alpha^3(1+\delta) = 0, \tag{18c}$$

$$\text{CR4:}\quad \delta(\beta - \phi\alpha)\ln\alpha\beta + \beta\ln\alpha - \phi\alpha\ln\beta = 0. \tag{18d}$$

These equations can be solved numerically. It is possible that for some values of ϕ multiple roots exist in the range $\alpha, \beta > 0$. For $\phi = 0.2$ and $\delta = \frac{10}{11}$, Eq. (18a) is satisfied by two positive values of β for α close to 1.0. However, for $\alpha > 1.2$, there is only one solution of Eq. (18a). Eq. (18b) has two solutions for β when $0 < \alpha < 1$, no admissible solution for $1 < \alpha < 1.8$ and only one solution for $\alpha > 1.8$. Eq. (18c) has a unique admissible solution for $0 < \alpha < 0.8$, and two admissible solutions for $0.8 < \alpha < 1.15$ and also for α greater than about 8.6.

2.3.4. Equal surface tractions on the edges

When equal surface tractions f_1, f_2 per unit current area are applied on the edges of the membrane, as will be in the case of pressure loading, then constitutive relations (2)–(4) require that $\beta = \alpha$ but constitutive relation (1) gives

$$\beta = \alpha, \quad \beta = \sqrt{\frac{1+2\delta}{1+\delta} - \alpha^2}. \tag{19}$$

Inequality (12b) must hold for constitutive relations (1)–(3).

2.3.5. Unequal surface tractions on the edges

For the case of $f_1 = \phi f_2$, $\phi < 1$, the relationships between β and α for the four constitutive relations are given below:

$$\text{CR1:} \quad \beta^4\phi(1+\delta) - \beta^2[\alpha^2\delta(-\phi+1) + \phi(1+2\delta)] - \alpha^2(\alpha^2(1+\delta) - (1+2\delta)) = 0, \tag{20a}$$

$$\text{CR2:} \quad \beta^2(\delta - \phi(1+\delta)) + \alpha^2(1+\delta(1-\phi)) + (\phi-1)(1+2\delta) = 0, \tag{20b}$$

$$\text{CR3:} \quad \beta^2(\delta(1-\phi)(2\alpha^2-1) + \alpha^2(1-\phi) - 1) + \alpha^2(\phi(1+\delta) - \delta) = 0, \tag{20c}$$

$$\text{CR4:} \quad \alpha^{(1+\delta)-\delta\phi} = \beta^{\phi(1+\delta)-\delta}. \tag{20d}$$

Whereas Eq. (20a) may have multiple solutions in the range $\delta, \beta > 0$, Eqs. (20b), (20c) and (20d) have a unique solution $\beta = \beta(\alpha) > 0$. Numerical computations reveal that for $\phi = 0.2$ and $\delta = \frac{10}{11}$, Eq. (20a) has two solutions for $0 < \alpha < 0.3815$ and $0.954 \leqslant \alpha < 1.2$, and only one solution for $\alpha > 1.2$. Eq. (20b) has a unique solution only when $0 < \alpha < 1.142$, and Eq. (20c) for $\alpha > 0.881$. However, Eq. (20d) has a unique solution for all positive values of α. The relationships between the values of α and β for the four constitutive relations are exhibited in Fig. 5.

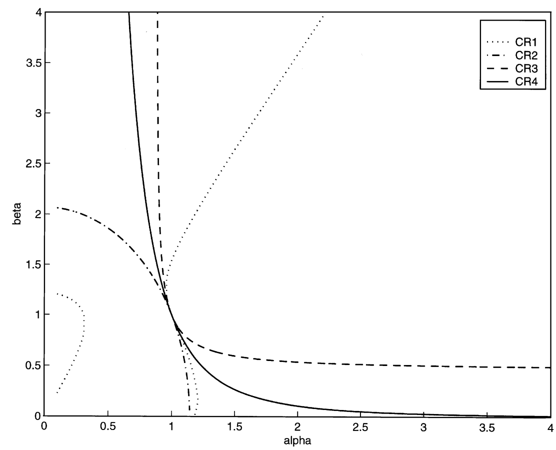

Fig. 5. The relationship between the two axial stretches, α and β, in a biaxially loaded membrane subjected to unequal surface tractions on the two edges.

2.4. Triaxial loading of a cube

We now consider a cube placed in a hard loading device and stretched by an axial stretch α in each direction. For the constitutive relations (1)–(4), the load F per unit undeformed area is given by

$$\bar{F}^{(1)} = \alpha(\alpha^2 - 1),$$
$$\bar{F}^{(2)} = \alpha^2(\alpha^2 - 1),$$
$$\bar{F}^{(3)} = (\alpha^2 - 1),$$
$$\bar{F}^{(4)} = \frac{2\ln\alpha}{\alpha}, \tag{21}$$

where $\bar{F} = 2F/(3\lambda + 2\mu)$. The stretch at an extreme load and hence at the initiation of a material instability and the corresponding non-dimensional axial nominal traction equal

$$\text{CR1:} \quad \alpha = 0.577,\ \bar{F} = -0.385,$$
$$\text{CR2:} \quad \alpha = 0.707,\ \bar{F} = -0.25,$$
$$\text{CR3:} \quad \text{always stable},$$
$$\text{CR4:} \quad \alpha = 2.718,\ \bar{F} = 0.736. \tag{22}$$

Constitutive relations (1), (2) and (4) predict material instability at universal stretches of 0.577, 0.707 and 2.718; the corresponding loads and deformations are compressive for constitutive relations (1) and (2) but tensile for constitutive relation (4).

For the case of equal nominal tractions, i.e., load per unit undeformed area, applied on the faces of the cube, each constitutive relation admits solutions $\alpha = \beta = \gamma$, $\alpha = \beta \neq \gamma$, $\beta = \gamma \neq \alpha$, $\gamma = \alpha \neq \beta$, and $\alpha \neq \beta \neq \gamma$, where α, β and γ are the axial stretches. Rivlin [7] has given such solutions for a cube made of a Mooney–Rivlin material.

For a cube subjected to a hydrostatic tension p, the stretch α must be same in every direction, and the pressure–stretch equations for the four constitutive relations (1)–(4) are

$$p^{(1)} = \left(\frac{3}{2}\lambda + \mu\right)\left(\frac{\alpha^2 - 1}{\alpha}\right), \tag{23a}$$

$$p^{(2)} = \left(\frac{3\lambda}{2} + \mu\right)(\alpha^2 - 1), \tag{23b}$$

$$p^{(3)} = \left(\frac{3\lambda}{2} + \mu\right)(1 - 1/\alpha^2), \tag{23c}$$

$$p^{(4)} = \left(\frac{3\lambda + 2\mu}{\alpha^3}\right)\ln \alpha. \tag{23d}$$

Fig. 6 depicts the normalized pressure versus the stretch for the four constitutive relations. For each one of the four equations it is a monotonically increasing function of α.

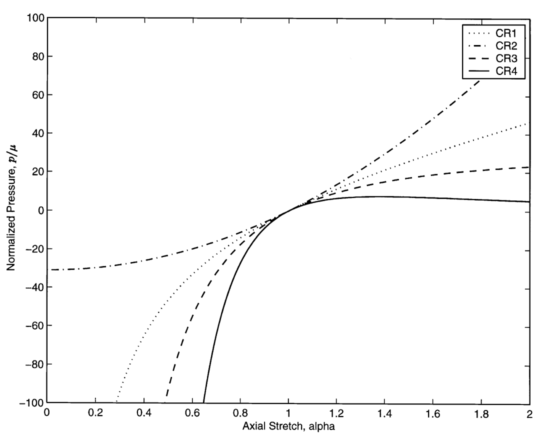

Fig. 6. The normalized pressure, p/μ, vs. the axial stretch, α, for a triaxially loaded cube.

3. Incompressible materials

For an incompressible material, a stress tensor equals the sum of two parts; one of these is not determined by the deformation gradient and the other is (e.g., see [2]). Analoges of constitutive relations (1)–(4) with the determinate part of the stress tensor linear in a measure of the deformation are

$$\mathbf{S} = -p(\mathbf{1} + 2\mathbf{E})^{-1} + 2\mu\mathbf{E}, \tag{24a}$$

$$\boldsymbol{\sigma} = -p\mathbf{1} + \mu(\mathbf{B} - \mathbf{1}), \tag{24b}$$

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2\mu\mathbf{A}, \tag{24c}$$

$$\bar{\mathbf{T}} = -p\mathbf{1} + 2\mu\mathbf{A}_{\ell}. \tag{24d}$$

The hydrostatic pressure p cannot be determined from the deformation field, but is completely determined by the balance of linear momentum and the traction boundary conditions prescribed either on a part or on the entire boundary of the body.

The deformation fields envisaged below are homogeneous. Thus, the determinate part of the stress tensor is constant throughout the body. The balance of linear momentum requires that the pressure field also be uniform.

3.1. Simple shear

For the deformation described by Eq. (6) the relationships between the shear stress T_{12} and the shear strain k for constitutive relations (1)–(4) are

$$T_{12}^{(1)} = \mu k(1 + k^2),$$

$$T_{12}^{(2)} = \mu k,$$

$$T_{12}^{(3)} = \mu k,$$

$$T_{12}^{(4)} = \frac{\mu}{\sqrt{1 + k^2/4}} \ln\left(1 + \frac{k^2}{2} + k\sqrt{1 + k^2/4}\right). \tag{25}$$

These relations are similar to Eqs. (7) for unconstrained materials. The pressure field, p, can be determined by requiring that any one of the planes $x_1 =$ const., $X_1 =$ const., and $x_2 = X_2 =$ const. be free of normal tractions. When normal tractions on planes $X_1 =$ const. (or $X_2 =$ const.) are taken to be zero, we get $T_{22} = 0$ (or $T_{11} = 0$) and there is no Poynting effect for each one of the four constitutive relations (24a)–(24d).

3.2. Simple extension

The axial load P per unit undeformed area is related as follows to the corresponding axial stretch α in an isotropic and homogeneous prismatic elastic body:

$$P^{(1)} = \mu\left(\alpha^3 - \alpha - \frac{1}{\alpha^3} + \frac{1}{\alpha^2}\right),$$

$$P^{(2)} = \mu\left(\alpha - \frac{1}{\alpha^2}\right),$$

$$P^{(3)} = \mu\left(1 - \frac{1}{\alpha^3}\right),$$

$$P^{(4)} = \frac{3\mu \ln \alpha}{\alpha}. \tag{26}$$

The dependence of the normalized axial load, P/μ, upon the axial stretch, α, is exhibited in Fig. 7 for the four constitutive relations. For $1 \leqslant \alpha \leqslant 2.5$, the load–stretch curves are close to each other for constitutive relations (24b)–(24d) and these are concave down, but that for the constitutive relation (24d) is concave upwards. Whereas $\mathrm{d}P/\mathrm{d}\alpha > 0$ for constitutive relations (24a)–(24c) it equals 0 at $\alpha = e = 2.718$ for the constitutive relation (24d) and the corresponding value of P is 1.1036μ.

3.3. Biaxial loading of a membrane

For loads given by (11a)–(11c) the pressure field can be determined from Eq. (11c). Relations between stretches α and β for the case of equal edge dead loads $F_1 = F_2$ are

$$\text{CR1:}\quad (\alpha - \beta)[1 - \alpha^2\beta^2 + \alpha^5\beta^5(\alpha^2 + \beta^2 + \alpha\beta - 1)] = 0, \tag{27a}$$

$$\text{CR2:}\quad (\alpha - \beta)[1 + \alpha^3\beta^3] = 0, \tag{27b}$$

$$\text{CR3:}\quad (\alpha - \beta)[\alpha^2 + \beta^2 + \alpha\beta - \alpha^4\beta^4] = 0, \tag{27c}$$

$$\text{CR4:}\quad \alpha \ln \beta^2\alpha - \beta \ln \alpha^2\beta = 0. \tag{27d}$$

These relations are universal in the sense that they do not depend upon the shear modulus μ. As for unconstrained materials $\alpha = \beta$ is a solution of each one of the Eqs. (27a)–(27d). Except for (27b) whose only solution is $\alpha = \beta$, other equations may also have solutions with $\beta \neq \alpha$ and $\alpha > 0$, $\beta > 0$. An attempt to seek numerical solutions of these equations revealed that only Eq. (27c) has a solution with $\alpha \neq \beta$; this is shown in Fig. 8. When multiple roots exist then the stable solution will minimize

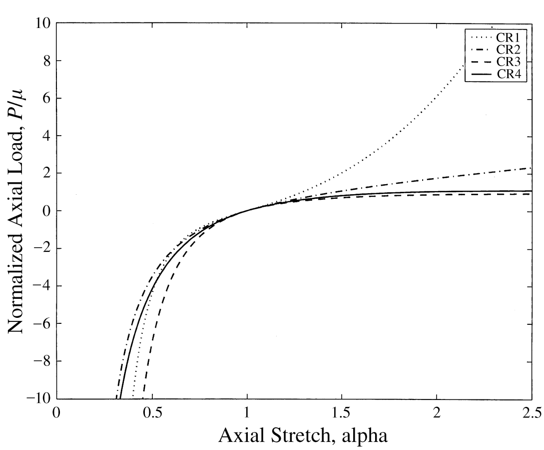

Fig. 7. The normalized axial load, P/μ, vs. the axial stretch, α, for simple extensional deformations of an elastic incompressible prismatic body.

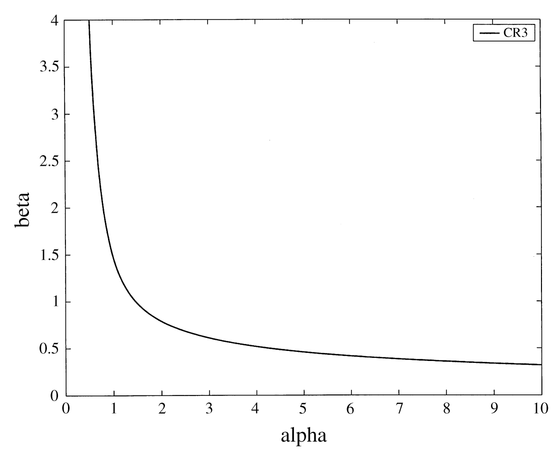

Fig. 8. The relationship between the two axial stretches, α and β, in a biaxially loaded incompressible elastic membrane subjected to equal edge dead loads.

either the stored energy or the potential energy depending upon whether stretches or dead loads are prescribed on the edges, respectively.

For a membrane placed in a hard loading device which applies equal stretches on all four edges, the relationships between the edge load F per unit undeformed length and the stretch α are

$$F^{(1)}/(\mu t_0) = (\alpha^2 - 1)[\alpha + (\alpha^2 + 1)/\alpha^9], \tag{28a}$$

$$F^{(2)}/(\mu t_0) = (\alpha - 1/\alpha^5), \tag{28b}$$

$$F^{(3)}/(\mu t_0) = (\alpha^3 - 1/\alpha^3), \tag{28c}$$

$$F^{(4)}/(\mu t_0) = 6(\ln \alpha)/\alpha. \tag{28d}$$

The load–deformation curves for the four constitutive relations are plotted in Fig. 9. For $\alpha > 1$, the F vs. α curves are concave upwards for the constitutive relations (24a) and (24c) and concave downwards for the other two relations. The load–deformation curves for the constitutive relations (24b) and (24d) are close to each other. For Eqs. (28a)–(28c) $\mathrm{d}F/\mathrm{d}\alpha > 0$. As for the case of an unconstrained material, the edge load for a membrane made of the material described by Eq. (24d) attains an extreme value at $\alpha = e = 2.718$. For

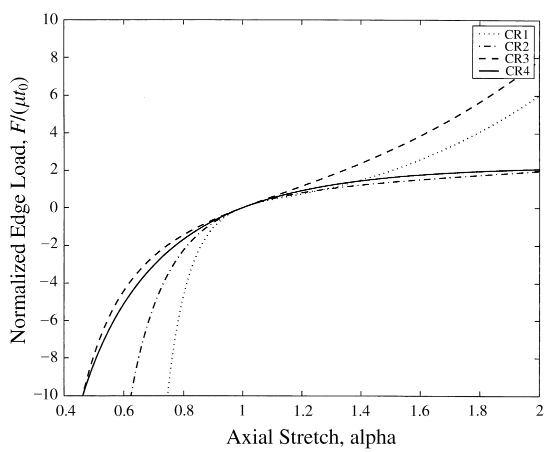

Fig. 9. The normalized edge load vs. the stretch in an incompressible elastic membrane stretched equally in all directions.

$F_1 = \phi F_2$, $\phi < 1$, the corresponding stretches in the two directions are related as follows:

$$\text{CR1:}\quad \left(\frac{1}{\alpha^2\beta^2} - 1\right)\left(\frac{\phi}{\alpha^2\beta^3} - \frac{1}{\alpha^3\beta^2}\right) + \alpha(\alpha^2 - 1) - \phi\beta(\beta^2 - 1) = 0, \tag{29a}$$

$$\text{CR2:}\quad \left(\frac{\phi}{\beta} - \frac{1}{\alpha}\right) + \alpha^2\beta^2(\alpha - \beta\phi) = 0, \tag{29b}$$

$$\text{CR3:}\quad \left(\frac{\phi}{\beta^3} - \frac{1}{\alpha^3}\right) - \alpha^2\beta^2\left(\frac{\phi}{\beta} - \frac{1}{\alpha}\right) = 0, \tag{29c}$$

$$\text{CR4:}\quad \alpha^{2\beta - \alpha\phi} = \beta^{2\alpha\phi - \beta}. \tag{29d}$$

It is possible that for some value of ϕ multiple roots exist in the range $\alpha, \beta > 0$. For $\phi = 0.2$, only Eq. (29c) could be satisfied by positive values of α and β and the relationship between the two is unique.

When equal surface tractions $f_1 = f_2$ per unit current area are applied on the edges of a membrane, then for constitutive relation (24b)–(24d) $\alpha = \beta$ is the only possible solution. However, for constitutive relation (24a), either $\beta = \alpha$ or $\beta = \sqrt{1 - \alpha^2}$. For the case of $f_1 = \phi f_2$, the relationships between β and α for the four constitutive relations are as follows:

$$\text{CR1:}\quad \frac{1}{\alpha^2\beta^2}(1 - \phi)\left(1 - \frac{1}{\alpha^2\beta^2}\right) + \alpha^2(\alpha^2 - 1) - \phi\beta^2(\beta^2 - 1) = 0, \tag{30a}$$

$$\text{CR2:}\quad \alpha^2 - \phi\beta^2 - \frac{1}{\alpha^2\beta^2}(1 - \phi) = 0, \tag{30b}$$

$$\text{CR3:}\quad \alpha^2\beta^2(1 - \phi) - \frac{1}{\alpha^2} + \frac{\phi}{\beta^2} = 0, \tag{30c}$$

$$\text{CR4:}\quad \alpha^{2 - \phi}\beta^{1 - 2\phi} = 1. \tag{30d}$$

The first three of these equations may have more than one solution $\beta = \beta(\alpha) > 0$, however, Eq. (30d) uniquely determines β in terms of α and ϕ.

3.4. Triaxial loading of a cube

For a cube made of an incompressible isotropic elastic material placed in a hard loading device that applies prescribed stretches on all faces of the cube, the hydrostatic pressure and hence the load deformation relation cannot be determined from constitutive relations (24a)–(24d). When either one or two pairs of opposite faces of the cube are kept traction free, the problems become similar to those treated above in Sections 3.2 and 3.3. For the case of equal dead loads (surface tractions per unit undeformed area) applied on all faces of the cube, each constitutive relation admits multiple solutions for the three axial stretches. A discussion of the stability of these solutions requires a knowledge of the stored energy function for each constitutive relation, and is not pursued here.

4. Discussion

Each one of the four constitutive relations (1)–(4) and (24a)–(24d) is consistent with the principles of Continuum Mechanics (e.g. see [2]) and reduces to Hooke's law for infinitesimal deformations.

The constitutive relation (3) gives a stable response for each one of the deformations studied. For simple extensional deformations, the axial load per unit undeformed area becomes infinite at an axial stretch of $\sqrt{\lambda/(3\lambda + 2\mu)}$ and is discontinuous at this value of the axial stretch. It predicts a tensile axial load for very large compressive axial deformations. According to the constitutive relation (4), the value of the deformation measure at the initiation of material instability is independent of material parameters.

Note that the discussion of structural instability also involves a consideration of the loading environment which is not accounted for herein, e.g. see [5]. The constitutive relation (1) predicts universal values of the instability stretch for simple extensional deformations of a prismatic body, biaxial stretching of a membrane, and triaxial stretching of a cube. Whereas constitutive relation (1) predicts instability for compressive deformations, (4) does so for tensile deformations. For simple shearing deformations, constitutive relation (1) predicts a hardening behavior.

For incompressible materials, constitutive relation (24d) predicts material instability in simple extensional deformations at an axial stretch of 2.718, in simple shearing deformations at a shear strain of 3.02, and in a biaxially stretched membrane at a stretch of 2.718 in each direction. In simple shearing deformations, the material described by constitutive relation (24a) exhibits hardening behavior. None of these constitutive relations predicts the Poynting effect. In simple extensional deformations, the axial load vs. the axial stretch curves for materials (24b) and (24d) are concave downwards, and the tangent modulus decreases with an increase in the deformation. However, for axial elongation of the bar made of material (24a), the tangent modulus increases with the elongation implying thereby that the material hardens as it is stretched.

One can conclude from the aforestated results for simple deformations that out of the four constitutive relations studied, only (2) and (3) for unconstrained materials and (24b) and (24c) for incompressible materials give results that agree at least qualitatively with *most* of the test observations detailed in Bell's encylopedia article [8]. If the material instability is a desired feature of a constitutive relation for finite deformations of an elastic material, then constitutive relation (3) is inadmissible.

As noted in the text, each one of the four constitutive relations studied reduces to Hooke's law for infinitesimal deformations. However, they predict quite different responses for moderate and large deformations.

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