Available online at www.sciencedirect.com
ScienceDirect
International Journal of Non-Linear Mechanics 43 (2008) 383–398
INTERNATIONAL JOURNAL OF NON-LINEAR MECHANICS
www.elsevier.com/locate/nlm

Exact solutions for radial deformations of a functionally graded isotropic and incompressible second-order elastic cylinder

R.C. Batra[a,*], G.L. Iaccarino[b]

[a]Department of Engineering Science and Mechanics, MC 0219, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA
[b]Department of Mathematics and Applications "Renato Caccioppoli", University of Naples Federico II, Via Cintia, 80126 Naples, Italy

Received 22 September 2007; received in revised form 11 January 2008; accepted 19 January 2008

Abstract

We find closed-form solutions for axisymmetric plane strain deformations of a functionally graded circular cylinder comprised of an isotropic and incompressible second-order elastic material with moduli varying only in the radial direction. Cylinder's inner and outer surfaces are loaded by hydrostatic pressures. These solutions are specialized to cases where only one of the two surfaces is loaded. It is found that for a linear through-the-thickness variation of the elastic moduli, the hoop stress for the first-order solution (or in a cylinder comprised of a linear elastic material) is a constant but that for the second-order solution varies through the thickness. The radial displacement, the radial stress and the hoop stress do not depend upon the second-order elastic constant but the hydrostatic pressure and hence the axial stress depends upon it. When the two elastic moduli vary as the radius raised to the power two or four, the radial and the hoop stresses in an infinite space with a pressurized cylindrical cavity equal the pressure in the cavity. For an affine variation of the elastic moduli, the hoop stress in an internally loaded cylinder made of a linear elastic isotropic and incompressible material at the point $R = \sqrt{R_{\text{in}} R_{\text{ou}}}$ is the same as that in a homogeneous cylinder. Here R_{in} and R_{ou} equal, respectively, the inner and the outer radius of the undeformed cylinder and R the radial coordinate of a point in the unstressed reference configuration.

Keywords: Analytical solutions; Functionally graded material; Circular cylinder; Second-order isotropic elastic material

1. Introduction

Functionally graded structures (FGSs) are macroscopically inhomogeneous bodies with material properties perceived to vary continuously in space even though they are generally comprised of more than one constituent. From a continuum point of view one ignores microstructural details and smears out the material moduli. Such materials abound in nature and common examples include a bamboo stick and human teeth. Engineered functionally graded materials (FGMs) include heat-resistant refractory shields with composition continuously varying from pure ceramic on one side to pure metal on the other side. It can potentially achieve multifunctional properties such as the thermal, wear, and oxidation resistance of ceramics combined with the high strength, high toughness, machinability, and bonding capability of metals. Another example is a nitrided steel.

Solutions of boundary-value problems for linear elastic and inhomogeneous materials are given in Lekhnitskii's book [1]. Batra [2] provided a numerical and an analytical solution to the problem of radial expansion of a cylinder made of an incompressible nonlinear (Mooney-Rivlin) elastic material with the two material parameters varying in the radial direction; he did not call the material functionally graded (FG). Even though FG polymeric and composite materials [3,4] were introduced in 1972, significant interest in FGMs seems to have originated with the first International Symposium on FGMs [5]. In view of the enormous literature on the subject, it is almost impossible to review it here. Accordingly, we only mention some of the analytical works. Vel and Batra [6,7] have provided analytical solutions to linear static and dynamic problems for FGSs. Horgan and Chan [8–10] have studied static deformations of cylinders, bars and rotating disks made of FGMs. Batra [11] has analyzed torsional deformations of a solid

* Corresponding author.
E-mail address: rbatra@vt.edu (R.C. Batra).

0020-7462/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijnonlinmec.2008.01.006

cylinder with material moduli varying continuously in the axial direction. Pan and Han [12] have provided analytical solutions for a simply supported flat magneto-electro-elastic plate loaded on its top surface. Cheng and Batra [13] have given a closed-form solution for a clamped elliptic FG plate, and for simply supported plates [14,15] have delineated correspondences between deflections and frequencies of FG and homogeneous plates. Jin and Batra [16,17] have studied fracture characteristics of FGSs under thermal and mechanical loads. Rather than assuming that the volume fraction of constituents varies through the thickness, Batra and Jin [18] proposed that a FG fiber-reinforced plate be fabricated by continuously varying the fiber orientation angle through the plate thickness. Vidoli and Batra [19] analyzed a piezoelectric rod with the orientation of the axis of polarization varying continuously in the cross-section. Batra and Love [20] have studied plane strain transient thermo-mechanical deformations of a thermo-visco-plastic prismatic body with material properties varying continuously with the distance of a point from the centroid of the rectangular cross-section.

Even though complicated non-linear problems involving FGMs can be solved numerically, e.g., see [2,20], their closed-form solutions can only be obtained for simple structural geometries and loadings. Here we analyze plane strain axisymmetric deformations of a FG cylinder loaded by internal and external pressures and comprised of an isotropic and incompressible second-order elastic material with material moduli varying only in the radial direction. Signorini's [21] perturbation method is adopted to analyze this problem. The reader is referred to Truesdell and Noll's [22] book for details of the Signorini method. Iaccarino et al. [23] have used Signorini's method to study axisymmetric deformations of a cylinder and a sphere with pressures applied to internal and external surfaces, and the material of each taken to be isotropic, homogeneous, unconstrained and second-order elastic. Saint-Venant's problems for second-order elastic materials have also been studied in [24–26]. Radial deformations of FG cylinders comprised of linear elastic materials have been studied in [27–37].

Here we analyze the cylinder problem for a second-order elastic FG material with material moduli varying continuously in the radial direction and the cylinder material to be isotropic and incompressible. Whereas there are six moduli in the constitutive relation for an isotropic second-order elastic compressible material, the second-order elastic incompressible material is characterized by two material parameters [22]. It is common to consider rubberlike, polymeric and biological soft tissue materials as incompressible.

For isotropic, unconstrained, homogeneous and linear elastic materials, the solution for a plane stress problem can be obtained from that for a plane strain problem by modifying Young's modulus and Poisson's ratio. However, such is not the case for plane problems for second-order elastic materials. Furthermore, plane strain problems for a linear elastic incompressible material cannot, in general, be analyzed by setting Poisson's ratio equal to 0.5 in the solution of the corresponding problem for a compressible linear elastic material.

The rest of the paper is organized as follows. Section 2 gives the problem formulation. Section 3 describes the solution for a cylinder made of a second-order elastic incompressible material subjected to internal and external pressures with the elastic moduli given by a power law. For the first-order problem, these results are specialized to cases when the applied pressure acts on only one of the surfaces. For an affine variation of the elastic moduli in the radial direction, displacements and stresses are given in Section 4. Solutions for the case of an infinite space with a pressurized cylindrical cavity at its center are deduced in Section 6. Section 7 describes conclusions of the work.

2. Problem formulation

We consider an infinitely long hollow cylinder of inner radius R_{in} and outer radius R_{ou} in the unstressed reference configuration. The cylinder, made of an isotropic second-order elastic material, is loaded by pressures p_{in} and p_{ou}, respectively, on its inner and outer surfaces as shown in Fig. 1. We assume that values of material parameters of the cylinder vary only in the radial direction. Since material properties, the cylinder geometry, and the applied loads are independent of the angular position and the axial coordinate of a point, we presume that its deformations are axisymmetric and are independent of the axial coordinate, z. Thus a material point of the cylinder moves only in the radial direction. Let r and R denote radial coordinates of a point in the present and the reference configurations, respectively, and $u(R)=r(R)-R$ its displacement in the radial direction. Note that the radial displacement of a point also induces strain in the circumferential direction, and the state of deformation in the cylinder is that of plane strain in the $r\theta$-plane where θ is the angular position of a point. Even though the axial strain identically vanishes, the axial stress is not necessarily zero. However, it is independent of the axial coordinate, z.

In cylindrical coordinates (r, θ, z) physical components of the displacement gradient $\mathbf{H}$ are given by

$$[H]=\begin{bmatrix} u' & 0 & 0 \\ 0 & \dfrac{u}{R} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.1}$$

where $u'=\mathrm{d}u/\mathrm{d}R$, and (R, Θ, Z) are coordinates of a point in the reference configuration. In order for the deformation (2.1) to occur in a body comprised of an incompressible material, it must be volume preserving. That is,

$$\det(\mathbf{1}+\mathbf{H})=1. \tag{2.2}$$

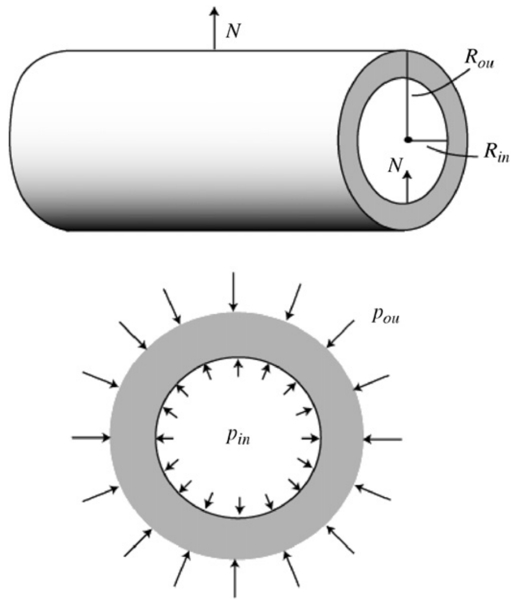

Fig. 1. A schematic sketch of the problem studied.

Expanding the left-hand side of Eq. (2.2) in $\mathbf{H}$ and retaining terms up to second-order in $\mathbf{H}$, we get

$$2I_{\mathbf{H}} + (I_{\mathbf{H}})^2 - I_{\mathbf{H}^2} = 0, \tag{2.3}$$

where $I_H = \mathrm{tr}(\mathbf{H})$ is the sum of the diagonal components of $\mathbf{H}$. Thus for a second-order elastic incompressible material, the displacement gradient $\mathbf{H}$ must satisfy Eq. (2.3).

The constitutive relation for a second-order elastic incompressible material is [22]

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2\mu\mathbf{E} + \mu\mathbf{H}\mathbf{H}^{\mathrm{T}} + \alpha\mathbf{E}^2, \tag{2.4}$$

where

$$\mathbf{E} = (\mathbf{H} + \mathbf{H}^{\mathrm{T}})/2 \tag{2.5}$$

is the infinitesimal strain tensor, $\boldsymbol{\sigma}$ the Cauchy stress, p the hydrostatic pressure not determined from the deformation field, $\mathbf{1}$ the identity tensor, $I_{\mathbf{E}} = \mathrm{tr}(\mathbf{E})$, μ the shear modulus, and α the second-order elastic constant of the material. Both μ and α have units of stress. For a FG cylinder μ and α are taken to be functions of the radial coordinate, R. In terms of the first Piola–Kirchhoff stress tensor $\mathbf{T}$, Eq. (2.4) becomes [22]

$$\mathbf{T} = -p(\mathbf{1} - \mathbf{H}^{\mathrm{T}} + \mathbf{H}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}) + 2\mu(\mathbf{E} - \mathbf{E}\mathbf{H}^{\mathrm{T}} + \mathbf{H}\mathbf{H}^{\mathrm{T}}/2) + \alpha\mathbf{E}^2. \tag{2.6}$$

Besides Eq. (2.3), equations governing static deformations of the body are

$$\begin{aligned} &\mathrm{Div}\,\mathbf{T} = \mathbf{0}, \quad R_{\mathrm{in}} < R < R_{\mathrm{ou}}, \\ &\mathbf{TN} = -p_{\mathrm{in}}\frac{r}{R_{\mathrm{in}}}\mathbf{N} \quad \text{on } R = R_{\mathrm{in}}, \quad \mathbf{TN} = -p_{\mathrm{ou}}\frac{r}{R_{\mathrm{ou}}}\mathbf{N} \quad \text{on } R = R_{\mathrm{ou}}. \end{aligned} \tag{2.7}$$

Here Div is the divergence operator with respect to coordinates in the reference configuration, $\mathbf{N}$ is an outward unit normal to the surface in the reference configuration, and p_{in} and p_{ou} are pressures (normal force per unit surface area in the present or the deformed configuration) acting, respectively, on the inner surface $R = R_{\mathrm{in}}$ and the outer surface $R = R_{\mathrm{ou}}$.

For a FG cylinder we assume that the material moduli μ and α depend upon the radial coordinate R as follows:

$$\mu = \mu_0 R^n, \quad \alpha = \alpha_0 R^n, \tag{2.8}$$

where μ_0, α_0 and n are constants. Whereas n is non-dimensional, units of μ_0 and α_0 depend upon those of R and the value of n. For a homogeneous material $n = 0$, and units of μ_0 and α_0 are those of stress. A solution of the first-order problem for an arbitrary variation of μ is given in [38].

The FGMs are generally made of two or more constituents and their effective elastic moduli are derived by using either the rule of mixtures or a homogenization technique (e.g., see [6]). For a non-linear elastic material homogenization methods are scarce, and a possibility is to employ numerical experiments [39] to deduce effective properties from those of the constituent phases. The assumption (2.8), similar to that adopted in [8–10], simplifies the algebraic work.

We non-dimensionalize stresses, α, and pressures by $\bar{\mu} = \mu_0 R_{\mathrm{ou}}^n$, displacements by R_{ou}, and the radial coordinate R by R_{ou}. Henceforth we use non-dimensional variables and denote them by the same symbols as before. When studying problems for an infinite space with a cylindrical cavity, we use R_{in} to non-dimensionalize R.

Let $\varepsilon = \max(p_{\mathrm{in}}, p_{\mathrm{ou}})$. For concreteness, let $p_{\mathrm{in}} > p_{\mathrm{ou}}$; thus $\varepsilon = p_{\mathrm{in}}$. Then boundary conditions given in Eqs. $(2.7)_2$ and $(2.7)_3$ can be written as

$$\mathbf{TN} = -\varepsilon \frac{r}{R_{\mathrm{in}}}\mathbf{N} \quad \text{on } R = R_{\mathrm{in}}, \quad \mathbf{TN} = -\frac{p_{\mathrm{ou}}}{p_{\mathrm{in}}}\varepsilon\frac{r}{R_{\mathrm{ou}}}\mathbf{N} \quad \text{on } R = R_{\mathrm{ou}}. \tag{2.9}$$

We assume that $\varepsilon \ll 1$, the pressure p and the radial displacement u are analytic functions of ε, expand them in terms of Taylor series, and retain terms up to second order in ε. Recalling that for $\varepsilon = 0$, $u = 0$ and $p = 0$, we have

$$\begin{aligned} p &= \varepsilon p^{(1)} + \varepsilon^2 p^{(2)}, \\ u &= \varepsilon u^{(1)} + \varepsilon^2 u^{(2)}. \end{aligned} \tag{2.10}$$

Up to second-order terms in ε, substitution from Eq. (2.10) into Eq. (2.6) gives

$$\mathbf{T} = \varepsilon \mathbf{T}^{(1)} + \varepsilon^2 \mathbf{T}^{(2)}, \tag{2.11}$$

where

$$\begin{aligned} \mathbf{T}^{(1)} &= -p^{(1)}\mathbf{1} + 2R^n\mathbf{E}^{(1)}, \\ \mathbf{T}^{(2)} &= -p^{(2)}\mathbf{1} + 2R^n\mathbf{E}^{(2)} - R^n\mathbf{H}^{(1)^{\mathrm{T}}}\mathbf{H}^{(1)^{\mathrm{T}}} + p^{(1)}\mathbf{H}^{(1)^{\mathrm{T}}} + \alpha_0 R^n \mathbf{E}^{(1)^2}, \end{aligned} \tag{2.12}$$

$$\mathbf{H}^{(i)} = \begin{bmatrix} \dfrac{\mathrm{d}u^{(i)}}{\mathrm{d}R} & 0 & 0 \\ 0 & \dfrac{u^{(i)}}{R} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad i = 1, 2, \tag{2.13}$$

$$2\mathbf{E}^{(i)} = \mathbf{H}^{(i)} + \mathbf{H}^{(i)^{\mathrm{T}}} \quad \text{etc.}$$

Substitution from Eq. $(2.10)_2$ into Eq. (2.3) gives

$$2\varepsilon\,\mathrm{tr}(\mathbf{H}^{(1)}) + \varepsilon^2[2\,\mathrm{tr}(\mathbf{H}^{(2)}) - \mathrm{tr}((\mathbf{H}^{(1)})^2)] = 0. \tag{2.14}$$

Equilibrium equations, boundary conditions, and equations preserving the volume of the body for the first- and the second-order problems can be written as

$$\begin{aligned} &\mathrm{Div}\,\mathbf{T}^{(1)} = \mathbf{0}, \quad R_{\mathrm{in}} < R < R_{\mathrm{ou}}, \\ &\mathbf{T}^{(1)}\mathbf{N} = -p_{\mathrm{in}}^{(1)}\mathbf{N} \quad \text{on } R = R_{\mathrm{in}}, \\ &\mathbf{T}^{(1)}\mathbf{N} = -p_{\mathrm{ou}}^{(1)}\mathbf{N} \quad \text{on } R = R_{\mathrm{ou}}, \\ &\mathrm{tr}(\mathbf{H}^{(1)}) = 0, \quad R_{\mathrm{in}} < R < R_{\mathrm{ou}} \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} &\mathrm{Div}\,\mathbf{T}^{(2)} = \mathbf{0}, \quad R_{\mathrm{in}} < R < R_{\mathrm{ou}}, \\ &\mathbf{T}^{(2)}\mathbf{N} = -p_{\mathrm{in}}^{(2)}\mathbf{N} \quad \text{on } R = R_{\mathrm{in}}, \\ &\mathbf{T}^{(2)}\mathbf{N} = -p_{\mathrm{ou}}^{(2)}\mathbf{N} \quad \text{on } R = R_{\mathrm{ou}}, \\ &2\,\mathrm{tr}(\mathbf{H}^{(2)}) - \mathrm{tr}((\mathbf{H}^{(1)})^2) = 0, \quad R_{\mathrm{in}} < R < R_{\mathrm{ou}}. \end{aligned} \tag{2.16}$$

Here

$$p_{\mathrm{in}}^{(1)} = 1, \quad p_{\mathrm{ou}}^{(1)} = p_{\mathrm{ou}}/p_{\mathrm{in}}, \tag{2.17}$$

and

$$p_{\text{in}}^{(2)}\mathbf{N} = p_{\text{in}}(\text{tr}(\mathbf{H}^{(1)})\mathbf{1} - (\mathbf{N}\cdot(\mathbf{E}^{(1)}\mathbf{N})))\mathbf{N} = p_{\text{in}}\frac{u^{(1)}(R_{\text{in}})}{R_{\text{in}}}\mathbf{N} \quad \text{on } R = R_{\text{in}},$$

$$p_{\text{ou}}^{(2)}\mathbf{N} = \left(\frac{p_{\text{ou}}}{p_{\text{in}}}\right)(\text{tr}(\mathbf{H}^{(1)})\mathbf{1} - (\mathbf{N}\cdot(\mathbf{E}^{(1)}\mathbf{N}))\mathbf{N}) = \frac{p_{\text{ou}}}{p_{\text{in}}}\frac{u^{(1)}(R_{\text{ou}})}{R_{\text{ou}}}\mathbf{N} \quad \text{on } R = R_{\text{ou}}. \tag{2.18}$$

Thus pressures to be applied on the inner and the outer surfaces of the cylinder for the second-order problem depend upon the solution of the first-order problem. Eq. (2.18) follows from Eq. $(24)_2$ of [23]. Note that for a given problem p_{in} and p_{ou} have assigned values, and do not vary with the deformation.

3. Analytical solution for power law variation of the two moduli

3.1. First-order problem ($n \neq 2, n \neq 4$)

We first study the problem when the exponent n in Eq. (2.8) is different from 2 and 4. Subsequently we provide solutions for $n = 2$ and 4.

Eqs. $(2.13)_1$ and $(2.15)_4$ yield

$$\frac{du^{(1)}}{dR} + \frac{u^{(1)}}{R} = 0, \tag{3.1}$$

whose solution is

$$u^{(1)} = B_1/R, \tag{3.2}$$

where B_1 is a constant. Substitution from Eq. (3.2) into Eqs. $(2.13)_1$ and $(2.13)_2$ and the result into Eq. $(2.12)_1$ gives the following for non-zero physical components of the first Piola–Kirchhoff stress tensor:

$$T_{RR}^{(1)} = -2B_1R^{-2+n} - p^{(1)}, \quad T_{\Theta\Theta}^{(1)} = 2B_1R^{-2+n} - p^{(1)}, \quad T_{ZZ}^{(1)} = -p^{(1)}. \tag{3.3}$$

Thus Eq. $(2.15)_1$ implies that the pressure $p^{(1)}$ is a function of R only, and

$$p^{(1)} = B_2 - \frac{2nB_1}{n-2}R^{-2+n}, \tag{3.4}$$

where B_2 is a constant of integration. Recall that $\mathbf{N} = (-1, 0, 0)$ on the surface $R = R_{\text{in}}$, and $\mathbf{N} = (1, 0, 0)$ on the surface $R = R_{\text{ou}}$. From boundary conditions $(2.15)_2$ and $(2.15)_3$ and Eqs. (3.3) and (3.4) we conclude that

$$B_1 = -\frac{(p_{\text{ou}} - p_{\text{in}})(n-2)}{4p_{\text{in}}(R_{\text{ou}}^{n-2} - R_{\text{in}}^{n-2})}, \quad B_2 = \frac{p_{\text{in}}R_{\text{ou}}^{n-2} - p_{\text{ou}}R_{\text{in}}^{n-2}}{p_{\text{in}}(R_{\text{ou}}^{n-2} - R_{\text{in}}^{n-2})}. \tag{3.5}$$

Note that

$$T_{\Theta\Theta}^{(1)} - T_{RR}^{(1)} = 4B_1R^{-2+n}, \tag{3.6}$$

and has the same sign as B_1 since $R^{-2+n} > 0$.

Substitution from Eqs. (3.4) and (3.5) into Eqs. (3.2) and (3.3) gives the following expressions for the radial displacement and the stresses:

$$u^{(1)} = -\frac{(p_{\text{ou}} - p_{\text{in}})(n-2)}{4p_{\text{in}}(R_{\text{ou}}^{n-2} - R_{\text{in}}^{n-2})}\frac{1}{R},$$

$$T_{RR}^{(1)} = \frac{p_{\text{ou}}(R_{\text{in}}^{n-2} - R^{n-2}) + p_{\text{in}}(R^{n-2} - R_{\text{ou}}^{n-2})}{(R_{\text{ou}}^{n-2} - R_{\text{in}}^{n-2})p_{\text{in}}},$$

$$T_{\Theta\Theta}^{(1)} = \frac{p_{\text{ou}}(-(n-1)R^{n-2} + R_{\text{in}}^{n-2}) + ((n-1)R^{n-2} - R_{\text{ou}}^{n-2})p_{\text{in}}}{(R_{\text{ou}}^{n-2} - R_{\text{in}}^{n-2})p_{\text{in}}},$$

$$T_{ZZ}^{(1)} = \frac{p_{\text{ou}}(0.5nR^{n-2} - R_{\text{in}}^{n-2}) + p_{\text{in}}(0.5nR^{n-2} - R_{\text{ou}}^{n-2})}{(R_{\text{ou}}^{n-2} - R_{\text{in}}^{n-2})p_{\text{in}}}. \tag{3.7}$$

It follows from Eqs. (2.10) and (2.11) that for the first-order problem and hence for a cylinder comprised of a linear elastic isotropic and incompressible FG material, values of radial displacement $u^{(1)}$ and stresses $\mathbf{T}^{(1)}$ equal $\varepsilon = p_{\text{in}}$ times their values listed in Eq. (3.7).

In the remainder of this sub-section, we drop the superscript (1) from various quantities to indicate their values for a linear elastic incompressible material.

It is clear from Eq. $(3.7)_3$ that for $n = 1$, the hoop stress is uniform throughout the cylinder thickness, and

$$T_{\Theta\Theta} = -\frac{p_{\text{ou}} R_{\text{ou}} - p_{\text{in}} R_{\text{in}}}{(R_{\text{ou}} - R_{\text{in}})}. \tag{3.8}$$

Thus $T_{\Theta\Theta} = 0$ throughout the cylinder thickness for $p_{\text{ou}} R_{\text{ou}} = p_{\text{in}} R_{\text{in}}$.

3.1.1. Solid cylinder

For a solid circular cylinder subjected to pressure p_{ou} on the outer surface, we set $\varepsilon = p_{\text{ou}}$, and conclude that the constant B_1 in Eq. (3.2) must vanish in order for the radial displacement to be finite at the center. Eqs. (3.2) and $(3.5)_2$ imply that $u = 0$ and $B_2 = 1$. Thus

$$T_{RR} = T_{\Theta\Theta} = T_{ZZ} = -p_{\text{ou}}.$$

The radial displacement of every point is zero, the cylinder does not deform, and the state of stress at a point is that of hydrostatic pressure.

3.1.2. Hollow cylinder subjected to internal pressure only

We now consider the case when a hollow cylinder is subjected to internal pressure only, i.e., $p_{\text{ou}} = 0$. Thus

$$T_{RR} = -p_{\text{in}} \frac{R^{n-2} - R_{\text{ou}}^{n-2}}{R_{\text{in}}^{n-2} - R_{\text{ou}}^{n-2}} < 0, \quad T_{\Theta\Theta} = p_{\text{in}} \frac{(1-n)R^{n-2} + R_{\text{ou}}^{n-2}}{R_{\text{in}}^{n-2} - R_{\text{ou}}^{n-2}}. \tag{3.9}$$

That is, the radial stress is everywhere compressive. For $n = 0$, i.e., a cylinder made of a homogeneous material, we recover classical expressions for stresses given in a book on linear elasticity even when the cylinder material is incompressible; e.g., see [40]. Note that most linear elasticity books consider the cylinder material to be compressible. From Eqs. (3.9) with $n = 0$, one can recover classical results for $R_{\text{ou}} \geqslant R_{\text{in}}$.

For $n = 1$,

$$T_{\Theta\Theta} = \frac{p_{\text{in}} R_{\text{in}}}{R_{\text{ou}} - R_{\text{in}}},$$

and is uniform throughout the cylinder. However, for a FG cylinder made of a compressible material studied in [8] the hoop stress for $n = 1$ varied through the cylinder thickness. Differentiation of both sides of Eq. $(3.9)_2$ with respect to R gives

$$\frac{\mathrm{d}T_{\Theta\Theta}}{\mathrm{d}R} = p_{\text{in}} \frac{(1-n)(n-2)R^{n-3}}{R_{\text{in}}^{n-2} - R_{\text{ou}}^{n-2}}. \tag{3.10}$$

Eq. (3.10) implies that for a homogeneous cylinder (i.e., $n = 0$) the hoop stress monotonically decreases from the inner radius to the outer radius. However, for a FG cylinder with $1 < n < 2$ and $n > 2$, $\mathrm{d}T_{\Theta\Theta}/\mathrm{d}R > 0$ throughout the cylinder thickness and it is negative for $n < 1$. Thus the variation of $T_{\Theta\Theta}$ through the cylinder thickness depends upon the value of n.

We now investigate the sign of $T_{\Theta\Theta}$ for $n \neq 1, 2, 4$, and first assume that $n > 2$. The function

$$f(R) = R^{n-2}, \quad R_{\text{in}} \leqslant R \leqslant R_{\text{ou}} \tag{3.11}$$

is positive and monotonically increasing. Its minimum and maximum values equal R_{in}^{n-2} and R_{ou}^{n-2}, respectively. It follows from Eq. $(3.9)_2$ that

$$T_{\Theta\Theta} \geqslant 0 \text{ if and only if } (1-n)R^{n-2} + R_{\text{ou}}^{n-2} \leqslant 0 \quad \text{or} \quad R^{n-2} \geqslant \frac{R_{\text{ou}}^{n-2}}{n-1}. \tag{3.12}$$

We define

$$\beta(n) = (n-1)^{1/(n-2)}, \tag{3.13}$$

and note that $1 < \beta(n) \leqslant 2$. Thus Eqs. $(3.9)_2$, (3.12) and (3.13) imply that for $n > 2$,

$$T_{\Theta\Theta} \geqslant 0 \quad \text{when } \frac{R_{\text{ou}}}{\beta(n)} \leqslant R < R_{\text{ou}},$$

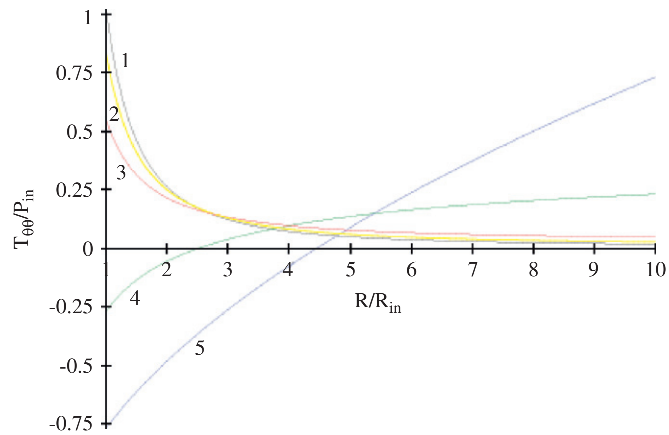

Fig. 2. For $R_{ou}/R_{in} = 10$ and different values of n, variation of the hoop stress through the thickness of the cylinder. Curves 1 (black line), 2 (yellow line), 3 (red line), 4 (green line), 5 (blue line) are for $n = 0$, 0.2, 0.5, 1.5 and 2.5, respectively.

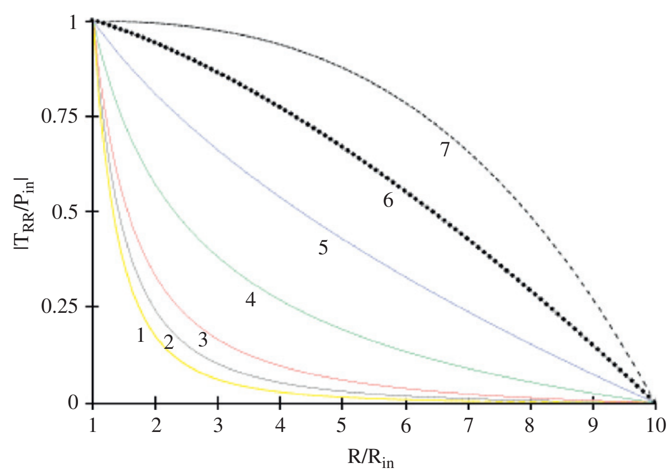

Fig. 3. For $R_{ou}/R_{in} = 10$ and different values of n, variation of hoop stress through the thickness of the cylinder. Curves 1 (yellow line), 2 (black line), 3 (red line), 4 (green line), 5 (blue line), 6 (dotdash), 7 (dash) are for $n = -0.5$, 0, 0.5, 1.5, 2.5, 3.5 and 5, respectively.

$$T_{\Theta\Theta} \leqslant 0 \quad \text{when } R_{in} < R \leqslant \frac{R_{ou}}{\beta(n)}, \tag{3.14}$$

and if $R_{ou}/\beta(n) \leqslant R_{in}$, then $T_{\Theta\Theta} \geqslant 0$ for all $R \in [R_{in}, R_{ou}]$.

For $n < 2$, $f(R)$ defined by Eq. (3.11) is a positive monotonically decreasing function of R, and its minimum and maximum values for $R_{in} \leqslant R \leqslant R_{ou}$ equal R_{ou}^{n-2} and R_{in}^{n-2}, respectively. Thus

$$T_{\Theta\Theta} \geqslant 0 \text{ if and only if } (1-n)R^{n-2} + R_{ou}^{n-2} \geqslant 0. \tag{3.15}$$

For $1 < n < 2$, the maximum value of $T_{\Theta\Theta}$ occurs at the outermost surface, and $T_{\Theta\Theta} \geqslant 0$ for $R_{in} \leqslant R \leqslant R_{ou}/\beta(n)$. For $n < 1$, $T_{\Theta\Theta}$ is always positive and has the maximum value at the innermost surface of the cylinder. Thus the inhomogeneity in the shear modulus strongly affects the through-the-thickness variation of the hoop stress.

For $R_{ou}/R_{in} = 10$ and different values of n we have plotted in Figs. 2 and 3 the variation with R/R_{in} of $T_{\Theta\Theta}/p_{in}$ and T_{RR}/p_{in}.

In order to show that one cannot obtain results for the present problem by setting Poisson's ratio $\nu = 0.5$ in the solution for the compressible material reported in [8], we set $\nu = 0.5$ in their Eq. (2.4) and the resulting value of Poisson's ratio in their Eq. (2.8) to obtain

$$u'' + \frac{(n+1)}{R}u' + (n-1)\frac{u}{R^2} = 0, \tag{3.16}$$

which is quite different from our Eq. (3.1). This is because the volume preserving condition governs the displacement field in an incompressible material but Eq. (3.16) is derived from the balance of linear momentum and the constitutive relations.

3.1.3. Hollow cylinder subjected to external pressure only

For a hollow cylinder loaded on the external surface only, $p_{\rm in}=0$, and we set $\varepsilon=p_{\rm ou}$. Eqs. (2.10) and (3.7) imply that

$$u=\frac{p_{\rm ou}(2-n)}{4(R_{\rm ou}^{n-2}-R_{\rm in}^{n-2})}\frac{1}{R},\quad T_{RR}=-\frac{p_{\rm ou}(R^{n-2}-R_{\rm in}^{n-2})}{(R_{\rm ou}^{n-2}-R_{\rm in}^{n-2})},$$

$$T_{\Theta\Theta}=\frac{p_{\rm ou}((1-n)R^{n-2}+R_{\rm in}^{n-2})}{(R_{\rm ou}^{n-2}-R_{\rm in}^{n-2})},\quad T_{ZZ}=\frac{-p_{\rm ou}(0.5nR^{n-2}-R_{\rm in}^{n-2})}{(R_{\rm ou}^{n-2}-R_{\rm in}^{n-2})}. \tag{3.17}$$

For $n=0$, i.e., a homogeneous cylinder, Eqs. $(3.17)_2$ and $(3.17)_3$ give

$$T_{RR}=-\frac{p_{\rm ou}R_{\rm ou}^2}{R_{\rm ou}^2-R_{\rm in}^2}\left(1-\frac{R_{\rm in}^2}{R^2}\right)\leqslant 0,\quad T_{\Theta\Theta}=-\frac{p_{\rm ou}R_{\rm ou}^2}{R_{\rm ou}^2-R_{\rm in}^2}\left(1+\frac{R_{\rm in}^2}{R^2}\right)<0, \tag{3.18}$$

which are the same as in a cylinder made of a compressible material. One can recover classical results for $R_{\rm ou}\gg R_{\rm in}$.

For $n>2$, $T_{RR}\leqslant 0$, and

$$T_{\Theta\Theta}\leqslant 0 \text{ if and only if } \left(1-(n-1)\left(\frac{R}{R_{\rm in}}\right)^{n-2}\right)\leqslant 0. \tag{3.19}$$

Thus $T_{\Theta\Theta}$ vanishes at the point $R=R_{\rm in}(1/(n-1))^{1/(n-2)}$ provided that it is within the cylinder. For $(1/(n-1))^{1/(n-2)}<R_{\rm ou}/R_{\rm in}$, or $R_{\rm ou}>2R_{\rm in}$, $T_{\Theta\Theta}$ vanishes at a point within the cylinder.

For $n<2$, $T_{RR}\leqslant 0$, and

$$T_{\Theta\Theta}\leqslant 0 \text{ if and only if } \left(1-(n-1)\left(\frac{R}{R_{\rm in}}\right)^{n-2}\right)\geqslant 0. \tag{3.20}$$

Differentiation with respect to R of both sides of Eq. $(3.17)_3$ gives

$$\frac{{\rm d}T_{\Theta\Theta}}{{\rm d}R}=p_{\rm ou}\frac{(1-n)(n-2)R^{n-3}}{R_{\rm ou}^{n-2}-R_{\rm in}^{n-2}}.$$

Whereas for a homogeneous cylinder with $n=0$, the hoop stress increases monotonically from the inner radius to the outer radius, for the FG cylinder its variation depends upon the value of n. For example, for $1<n<2$ and $n>2$, $T_{\Theta\Theta}$ decreases monotonically from the inner to the outer radius but the reverse holds for $n<1$.

3.2. Second-order problem ($n\neq 2$, $n\neq 4$)

Eqs. (3.2), $(2.13)_1$ and $(2.16)_4$ give

$$\frac{{\rm d}u^{(2)}}{{\rm d}R}+\frac{u^{(2)}}{R}-\frac{B_1^2}{R^4}=0, \tag{3.21}$$

whose solution is

$$u^{(2)}=\frac{B_3}{R}-\frac{B_1^2}{2R^3}, \tag{3.22}$$

where B_3 is a constant of integration. Substitution from Eq. (3.22) into Eq. (2.13) and the result into Eq. $(2.12)_2$ gives

$$T_{RR}^{(2)}=-2B_3R^{-2+n}+B_1^2R^{-4+n}\left(\frac{4(n-1)+\alpha_0(n-2)}{n-2}\right)-\frac{B_1B_2}{R^2}-p^{(2)},$$

$$T_{\Theta\Theta}^{(2)}=2B_3R^{-2+n}+B_1^2R^{-4+n}\left(\frac{-4(n-1)+\alpha_0(n-2)}{n-2}\right)+\frac{B_1B_2}{R^2}-p^{(2)},$$

$$T_{ZZ}^{(2)}=-p^{(2)}. \tag{3.23}$$

Substitution for stresses from Eq. (3.23) into the equilibrium equation $(2.16)_1$ for the second-order problem gives equations that imply that $p^{(2)}$ is a function of R only, and

$$p^{(2)}(R)=B_4+\frac{B_1^2[4(n-1)+\alpha_0(n-4)]}{(n-4)}R^{n-4}-\frac{2nB_3}{(n-2)}R^{n-2}, \tag{3.24}$$

where B_4 is a constant of integration. In order to find constants B_3 and B_4 from boundary conditions, we note from Eqs. (3.2), (2.13) and (2.18), that

$$p_{\rm in}^{(2)} = \frac{B_1}{R_{\rm in}^2}, \quad p_{\rm ou}^{(2)} = \frac{B_1 p_{\rm ou}}{R_{\rm ou}^2 p_{\rm in}}. \tag{3.25}$$

Thus from Eqs. $(2.16)_2$, $(2.16)_3$, $(3.23)_1$ and (3.24) we obtain

$$B_3 = \frac{3(n-2)^3(p_{\rm ou}-p_{\rm in})^2 R_{\rm ou}^2 R_{\rm in}^2 (R_{\rm ou}^4 R_{\rm in}^n - R_{\rm ou}^n R_{\rm in}^4)}{16(n-4)p_{\rm in}^2 (R_{\rm ou}^2 R_{\rm in}^n - R_{\rm in}^2 R_{\rm ou}^n)^3}, \tag{3.26}$$

$$B_4 = \frac{3(n-2)^2(p_{\rm ou}-p_{\rm in})^2 R_{\rm ou}^{n+2} R_{\rm in}^{n+2} (R_{\rm ou}^2 - R_{\rm in}^2)}{4(n-4)p_{\rm in}^2 (R_{\rm ou}^2 R_{\rm in}^n - R_{\rm in}^2 R_{\rm ou}^n)^3}. \tag{3.27}$$

3.3. Complete solution ($n \neq 2$, $n \neq 4$)

The complete solution of the second-order problem in terms of non-dimensional variables, for $n \neq 2, 4$, obtained by substituting for $u^{(1)}$, $u^{(2)}$, $p^{(1)}$, $p^{(2)}$, $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$ in Eqs. (2.10) and (2.11) is

$$u(R) = \frac{(n-2)R_{\rm in}^2 R_{\rm ou}^2 (p_{\rm in}-p_{\rm ou})}{4(R_{\rm ou}^n R_{\rm in}^2 - R_{\rm in}^n R_{\rm ou}^2)}\frac{1}{R} + \frac{(p_{\rm ou}-p_{\rm in})^2 (n-2)^2 R_{\rm in}^2 R_{\rm ou}^2}{16R(R_{\rm ou}^2 R_{\rm in}^n - R_{\rm in}^2 R_{\rm ou}^n)^2}\left[\frac{3(n-2)(R_{\rm ou}^4 R_{\rm in}^n - R_{\rm ou}^n R_{\rm in}^4)}{(n-4)(R_{\rm ou}^2 R_{\rm in}^n - R_{\rm in}^2 R_{\rm ou}^n)} - \frac{R_{\rm ou}^2 R_{\rm in}^2}{2R^2}\right], \tag{3.28}$$

$$\begin{aligned} p(R) = {} & \frac{R_{\rm ou}^2 R_{\rm in}^2}{(R_{\rm ou}^n R_{\rm in}^2 - R_{\rm in}^n R_{\rm ou}^2)}\left[(p_{\rm in} R_{\rm ou}^{n-2} - p_{\rm ou} R_{\rm in}^{n-2}) + \frac{n}{2}(p_{\rm ou}-p_{\rm in})R^{n-2}\right] + \frac{3(n-2)^2(p_{\rm ou}-p_{\rm in})^2 R_{\rm ou}^2 R_{\rm in}^2}{4(R_{\rm ou}^2 R_{\rm in}^n - R_{\rm in}^2 R_{\rm ou}^n)^2 (n-4)} \\ & \times \left[\frac{R_{\rm ou}^n R_{\rm in}^n (R_{\rm ou}^2 - R_{\rm in}^2)}{(R_{\rm ou}^2 R_{\rm in}^n - R_{\rm in}^2 R_{\rm ou}^n)} - \frac{n(R_{\rm ou}^4 R_{\rm in}^n - R_{\rm in}^4 R_{\rm ou}^n)}{2(R_{\rm ou}^2 R_{\rm in}^n - R_{\rm in}^2 R_{\rm ou}^n)}R^{n-2} + \frac{(4(n-1)+\alpha_0(n-4))R_{\rm ou}^2 R_{\rm in}^2}{12}R^{n-4}\right], \end{aligned}$$

$$T_{RR} = \frac{p_{\rm ou}(R_{\rm in}^{n-2} - R^{n-2}) + p_{\rm in}(R^{n-2} - R_{\rm ou}^{n-2})}{R_{\rm ou}^{n-2} - R_{\rm in}^{n-2}} + [c_{10} + c_{11}R^{-2} + c_{12}R^{n-2} + c_{13}R^{n-4}],$$

$$T_{\Theta\Theta} = \frac{p_{\rm ou}(-(n-1)R^{n-2} + R_{\rm in}^{n-2}) + ((n-1)R^{n-2} - R_{\rm ou}^{n-2})p_{\rm in}}{(R_{\rm ou}^{n-2} - R_{\rm in}^{n-2})} + [c_{20} + c_{21}R^{-2} + c_{22}R^{n-2} + c_{23}R^{n-4}], \tag{3.29}$$

where

$$c_{10} = \frac{3(-2+n)^2(p_{\rm ou}-p_{\rm in})^2 R_{\rm ou}^{2+n} R_{\rm in}^{2+n} (R_{\rm ou}^2 - R_{\rm in}^2)}{4(-4+n)c_{33}^3},$$

$$c_{33} = (R_{\rm ou}^n R_{\rm in}^2 - R_{\rm ou}^2 R_{\rm in}^n),$$

$$c_{11} = -\frac{(-2+n)(p_{\rm ou}-p_{\rm in})R_{\rm ou}^2 R_{\rm in}^2 (-p_{\rm in} R_{\rm ou} R_{\rm in}^2 + p_{\rm ou} R_{\rm ou}^2 R_{\rm in}^2)}{4c_{33}^2},$$

$$c_{12} = \frac{3(-2+n)^2(p_{\rm ou}-p_{\rm in})^2 R_{\rm ou}^2 R_{\rm in}^2 (-R_{\rm ou}^n R_{\rm in}^4 + R_{\rm ou}^4 R_{\rm in}^n)}{4(4-n)c_{33}^3},$$

$$c_{13} = -\frac{(-2+n)(-1+n)(p_{\rm ou}-p_{\rm in})^2 R_{\rm ou}^4 R_{\rm in}^4}{2(-4+n)c_{33}^2},$$

$$c_{20} = \frac{3(-2+n)^2(p_{\rm ou}-p_{\rm in})^2 R_{\rm ou}^{2+n} R_{\rm in}^{2+n} (R_{\rm ou}^2 - R_{\rm in}^2)}{4(-4+n)c_{33}^3},$$

$$c_{21} = \frac{(-2+n)(p_{\rm ou}-p_{\rm in})R_{\rm ou}^2 R_{\rm in}^2 (-p_{\rm in} R_{\rm ou}^n R_{\rm in}^2 + p_{\rm ou} R_{\rm ou}^2 R_{\rm in}^n)}{4c_{33}^2},$$

$$c_{22} = \frac{3(-2+n)^2(-1+n)(p_{\rm ou}-p_{\rm in})^2 R_{\rm ou}^2 R_{\rm in}^2 (-R_{\rm ou}^n R_{\rm in}^4 + R_{\rm ou}^4 R_{\rm in}^n)}{4(4-n)c_{33}^3},$$

$$c_{23}=\frac{(-2+n)(3-4n+n^2)(p_{ou}-p_{in})^2R_{ou}^4R_{in}^4}{2(4-n)c_{33}^2}. \tag{3.30}$$

Note that terms in brackets in Eqs. $(3.29)_2$ and $(3.29)_3$ are contributions from the second-order solution, and the second-order elastic constant α_0 appears only in the pressure field but not in the expressions for the radial and the hoop stresses. Whereas the first-order stresses are independent of the elastic moduli, μ_0 appears in the expressions for the second-order stresses. Even though stresses (except for the axial stress) and the radial displacement do not explicitly depend upon α_0, their values for the first-order and the second-order elastic materials are different. At any point, the difference between the two displacements is proportional to $(p_{ou}-p_{in})^2$ thereby necessitating a rather sophisticated sensor to detect it.

For $n=1$, $c_{22}=c_{23}=0$, but $c_{21}\neq 0$. Thus the hoop stress in the second-order elastic material is not constant throughout the thickness of the cylinder.

Recalling that the axial force, F_{ax}, acting on a cross-section of the cylinder is given by

$$F_{ax}=2\pi\int_{R_{in}}^{R_{ou}}T_{ZZ}R\,\mathrm{d}R, \tag{3.31}$$

the difference in the values of F_{ax} for the first-order and the second-order elastic materials is proportional to $(p_{ou}-p_{in})^2$. When measured accurately, it can be used to estimate the value of the second-order elastic constant α_0.

For a homogeneous second-order elastic material, $n=0$, and Eqs. (3.28) and (3.29) reduce to

$$\begin{aligned}
u(R)&=\frac{R_{in}^2R_{ou}^2(p_{ou}-p_{in})}{2(R_{in}^2-R_{ou}^2)}\frac{1}{R}+\frac{(p_{ou}-p_{in})^2R_{ou}^2R_{in}^2}{4R(R_{ou}^2-R_{in}^2)^2}\left(\frac{3}{2}(R_{ou}^2+R_{in}^2)-\frac{R_{ou}^2R_{in}^2}{2R^2}\right),\\
p(R)&=\frac{p_{in}R_{in}^2-p_{ou}R_{ou}^2}{R_{in}^2-R_{ou}^2}-\frac{3}{4}\frac{(p_{ou}-p_{in})^2R_{ou}^2R_{in}^2}{(R_{ou}^2-R_{in}^2)}\left(1-\frac{(1+\alpha_0)}{3}\frac{R_{ou}^2R_{in}^2}{R^4}\right),\\
T_{RR}&=\frac{p_{in}R_{in}^2(R^2-R_{ou}^2)+p_{ou}R_{ou}^2(-R^2+R_{in}^2)}{R^2(R_{ou}^2-R_{in}^2)}+d_{10}+d_{11}R^{-2}+d_{12}R^{-4},\\
T_{\Theta\Theta}&=\frac{p_{in}R_{in}^2(R^2+R_{ou}^2)-p_{ou}R_{ou}^2(R^2+R_{in}^2)}{R^2(R_{ou}^2-R_{in}^2)}+d_{10}-d_{11}R^{-2}-3d_{12}R^{-4},
\end{aligned} \tag{3.32}$$

where

$$\begin{aligned}
d_{10}&=\frac{3(p_{ou}-p_{in})^2R_{ou}^2R_{in}^2}{4(R_{ou}^2-R_{in}^2)^2},\\
d_{11}&=-\frac{R_{ou}^2R_{in}^2[-4p_{ou}p_{in}(R_{ou}^2+R_{in}^2)+p_{in}^2(3R_{ou}^2+R_{in}^2)+p_{ou}^2(R_{ou}^2+3R_{in}^2)]}{4(R_{ou}^2-R_{in}^2)^2},\\
d_{12}&=\frac{(p_{ou}-p_{in})^2R_{ou}^4R_{in}^4}{4(R_{ou}^2-R_{in}^2)^2}.
\end{aligned} \tag{3.33}$$

3.3.1. Homogeneous hollow cylinder subjected only to internal pressure

For $p_{ou}=0$, Eqs. $(3.32)_3$ and $(3.32)_4$ simplify to

$$\begin{aligned}
\frac{T_{RR}}{p_{in}}&=\frac{R_{in}^2}{R_{ou}^2-R_{in}^2}\left(1-\frac{R_{ou}^2}{R^2}\right)+\frac{p_{in}}{4}\frac{R_{ou}^2R_{in}^2}{(R_{ou}^2-R_{in}^2)^2}\left(3-\frac{3R_{ou}^2+R_{in}^2}{R^2}+\frac{R_{ou}^2R_{in}^2}{R^4}\right),\\
\frac{T_{\Theta\Theta}}{p_{in}}&=\frac{R_{in}^2}{R_{ou}^2-R_{in}^2}\left(1+\frac{R_{ou}^2}{R^2}\right)+\frac{p_{in}}{4}\frac{R_{ou}^2R_{in}^2}{(R_{ou}^2-R_{in}^2)^2}\left(3+\frac{3R_{ou}^2+R_{in}^2}{R^2}-\frac{3R_{ou}^2R_{in}^2}{R^4}\right).
\end{aligned} \tag{3.34}$$

For $R_{ou}\gg R_{in}$, Eqs. (3.34) yield

$$\frac{T_{RR}}{p_{in}}=-\frac{R_{in}^2}{R^2}+\frac{p_{in}R_{in}^2}{4}\left(-\frac{3}{R^2}+\frac{R_{in}^2}{R^4}\right),\qquad \frac{T_{\Theta\Theta}}{p_{in}}=\frac{R_{in}^2}{R^2}+\frac{3p_{in}R_{in}^2}{4}\left(\frac{1}{R^2}-\frac{R_{in}^2}{R^4}\right). \tag{3.35}$$

Eqs. (3.35) give the stress field in a uniformly pressurized cavity in an infinite homogeneous isotropic second-order elastic material. Second terms on the right-hand sides of Eqs. $(3.35)_1$ and $(3.35)_2$ represent corrections to stress fields in a linear elastic material.

Note that $T_{\Theta\Theta}$ on the inner surface is unaffected by the consideration of second-order effects. However, at inner points of the cylinder, $T_{\Theta\Theta}$ for the second-order elastic material is more than that for the linear elastic material.

3.3.2. Homogeneous hollow cylinder subjected only to external pressure

Setting $p_{\text{in}} = 0$ in Eqs. $(3.32)_3$ and $(3.32)_4$, we get

$$\frac{T_{RR}}{p_{\text{ou}}} = \frac{R_{\text{ou}}^2}{R_{\text{ou}}^2 - R_{\text{in}}^2}\left(-1 + \frac{R_{\text{in}}^2}{R^2}\right) + \frac{p_{\text{ou}}}{4}\frac{R_{\text{ou}}^2 R_{\text{in}}^2}{(R_{\text{ou}}^2 - R_{\text{in}}^2)^2}\left(3 - \frac{R_{\text{ou}}^2 + 3R_{\text{in}}^2}{R^2} + \frac{R_{\text{ou}}^2 R_{\text{in}}^2}{R^4}\right),$$

$$\frac{T_{\Theta\Theta}}{p_{\text{ou}}} = -\frac{R_{\text{ou}}^2}{R_{\text{ou}}^2 - R_{\text{in}}^2}\left(1 + \frac{R_{\text{in}}^2}{R^2}\right) + \frac{p_{\text{ou}}}{4}\frac{R_{\text{ou}}^2 R_{\text{in}}^2}{(R_{\text{ou}}^2 - R_{\text{in}}^2)^2}\left(3 + \frac{R_{\text{ou}}^2 + 3R_{\text{in}}^2}{R^2} - \frac{3R_{\text{ou}}^2 R_{\text{in}}^2}{R^4}\right). \tag{3.36}$$

For $R_{\text{ou}} \gg R_{\text{in}}$, we get the following from Eqs. (3.36):

$$\frac{T_{RR}}{p_{\text{ou}}} = \left(-1 + \frac{R_{\text{in}}^2}{R^2}\right) + \frac{p_{\text{ou}} R_{\text{in}}^2}{4}\left(-\frac{1}{R^2} + \frac{R_{\text{in}}^2}{R^4}\right), \quad \frac{T_{\Theta\Theta}}{p_{\text{ou}}} = -\left(1 + \frac{R_{\text{in}}^2}{R^2}\right) + \frac{p_{\text{ou}} R_{\text{in}}^2}{4}\left(\frac{1}{R^2} - \frac{3R_{\text{in}}^2}{R^4}\right). \tag{3.37}$$

Eqs. $(3.37)_1$ and $(3.37)_2$ give stresses in an infinite homogeneous medium containing a cylindrical cavity and subjected to uniform normal tractions at infinity. We conclude from Eq. $(3.37)_2$ that

$$\frac{T_{\Theta\Theta}}{p_{\text{ou}}}(R_{\text{in}}) = -2 - \frac{p_{\text{ou}}}{2}. \tag{3.38}$$

That is the magnitude of the hoop stress at the surface of the cylindrical cavity is slightly enhanced by the consideration of second-order effects.

3.4. The case $n = 2$

For $n = 2$ in Eq. (2.8), Eqs. (3.2) and (3.3) still hold with constants B_1 and B_2 replaced by $\bar{B}_1$ and $\bar{B}_2$ respectively, and Eqs. (3.4) and (3.5) become

$$p^{(1)} = \bar{B}_2 - 4\bar{B}_1 \ln R, \tag{3.39}$$

$$\bar{B}_1 = -\frac{(p_{\text{ou}} - p_{\text{in}})}{4p_{\text{in}} \ln(R_{\text{ou}}/R_{\text{in}})}, \quad \bar{B}_2 = -\frac{p_{\text{ou}}[2\ln(R_{\text{in}}) - 1] - p_{\text{in}}[2\ln(R_{\text{ou}}) - 1]}{2p_{\text{in}} \ln(R_{\text{ou}}/R_{\text{in}})}. \tag{3.40}$$

The radial displacement $u^{(2)}$ and stresses $\mathbf{T}^{(2)}$ for the second-order problem are given by Eqs. (3.22) and (3.23), respectively, with constants B_1 and B_3 replaced by $\bar{B}_1$ and $\bar{B}_3$, respectively. The pressure field (3.24) is now given by

$$p^{(2)}(R) = \bar{B}_4 - \frac{\bar{B}_1^2(2 - \alpha_0)}{R^2} - 4\bar{B}_3 \ln R, \tag{3.41}$$

where constants $\bar{B}_3$ and $\bar{B}_4$, determined from the boundary conditions, have the following expressions:

$$\bar{B}_3 = \frac{3(p_{\text{ou}} - p_{\text{in}})^2(R_{\text{ou}}^2 - R_{\text{in}}^2)}{32p_{\text{in}}^2 R_{\text{ou}}^2 R_{\text{in}}^2[\ln(R_{\text{ou}}/R_{\text{in}})]^3}, \quad \bar{B}_4 = \frac{3(p_{\text{ou}} - p_{\text{in}})^2[(2\ln(R_{\text{ou}}) - 1)R_{\text{ou}}^2 - (2\ln(R_{\text{in}}) - 1)R_{\text{in}}^2]}{16p_{\text{in}}^2 R_{\text{ou}}^2 R_{\text{in}}^2[\ln(R_{\text{ou}}/R_{\text{in}})]^3}. \tag{3.42}$$

Thus the complete solution of the problem for $n = 2$ is

$$u(R) = -\frac{(p_{\text{ou}} - p_{\text{in}})}{4\ln(R_{\text{ou}}/R_{\text{in}})}\frac{1}{R} + \frac{(p_{\text{ou}} - p_{\text{in}})^2}{32[\ln(R_{\text{ou}}/R_{\text{in}})]^3 R}\left[\frac{3(R_{\text{ou}}^2 - R_{\text{in}}^2)}{R_{\text{ou}}^2 R_{\text{in}}^2 \ln(R_{\text{ou}}/R_{\text{in}})} - \frac{1}{R^2}\right],$$

$$p(R) = \frac{-p_{\text{ou}}[2\ln(R_{\text{in}}) - 1] + p_{\text{in}}[2\ln(R_{\text{ou}}) - 1]}{2\ln(R_{\text{ou}}/R_{\text{in}})} + \frac{(p_{\text{ou}} - p_{\text{in}})}{\ln(R_{\text{ou}}/R_{\text{in}})}\ln R + \frac{3(p_{\text{ou}} - p_{\text{in}})^2}{8R_{\text{ou}}^2 R_{\text{in}}^2[\ln(R_{\text{ou}}/R_{\text{in}})]^3}$$
$$\times\left[(\ln(R_{\text{ou}}) - 0.5)R_{\text{ou}}^2 - (\ln(R_{\text{in}}) - 0.5)R_{\text{in}}^2 - \frac{(2 - \alpha_0)}{6}\frac{R_{\text{ou}}^2 R_{\text{in}}^2 \ln(R_{\text{ou}}/R_{\text{in}})}{R^2} - (R_{\text{ou}}^2 - R_{\text{in}}^2)\ln R\right],$$

$$T_{RR}=\frac{p_{\text{ou}}\ln(R_{\text{in}}/R)-p_{\text{in}}\ln(R_{\text{ou}}/R)}{\ln(R_{\text{ou}}/R_{\text{in}})}+e_{10}(R)+e_{11}(R)/R^2,$$

$$T_{\Theta\Theta}=\frac{p_{\text{ou}}(-1+\ln(R_{\text{in}}/R)+p_{\text{in}}(1-\ln(R_{\text{ou}}/R))}{\ln(R_{\text{ou}}/R_{\text{in}})}+e_{20}(R)+e_{21}(R)/R^2, \tag{3.43}$$

where

$$e_{10}(R)=\frac{3(p_{\text{ou}}-p_{\text{in}})^2[-R_{\text{ou}}^2\ln(R_{\text{ou}}/R)+R_{\text{in}}^2\ln(R_{\text{in}}/R)]}{8(\ln(R_{\text{ou}}/R_{\text{in}}))^3R_{\text{ou}}^2R_{\text{in}}^2},$$

$$e_{11}(R)=\frac{(p_{\text{ou}}-p_{\text{in}})[(3-2\ln(R_{\text{in}}/R))p_{\text{ou}}+p_{\text{in}}(-3+2\ln(R_{\text{ou}}/R))]}{8(\ln(R_{\text{ou}}/R_{\text{in}}))^2},$$

$$e_{20}(R)=\frac{3(p_{\text{ou}}-p_{\text{in}})^2[(1-\ln(R_{\text{ou}}/R))R_{\text{ou}}^2+(-1+\ln(R_{\text{in}}/R))R_{\text{in}}^2]}{8(\ln(R_{\text{ou}}/R_{\text{in}}))^2},$$

$$e_{21}(R)=-\frac{(p_{\text{ou}}-p_{\text{in}}[(1+2\ln(R/R_{\text{in}})p_{\text{ou}}-(1+2\ln(R/R_{\text{ou}})p_{\text{in}}]}{8(\ln(R_{\text{ou}}/R_{\text{in}}))^2}. \tag{3.44}$$

As for the problem studied in Section 3.3, only the pressure depends upon α_0. Stresses in this case have more complicated distributions through the thickness as compared to that when $n \neq 2, 4$.

For $n=2$, the through-the-thickness distributions of the radial and the hoop stresses in a linear elastic FG cylinder are similar to those for $n=\frac{3}{2}$ and their plots are thus omitted.

3.5. The case $n=4$

For $n=4$ in Eq. (2.8), Eqs. (3.2) and (3.3) hold with constant B_1 replaced by $\tilde{B}_1$, and Eq. (3.4) holds with B_2 and B_1 replaced by $\tilde{B}_2$ and $\tilde{B}_1$, respectively. That is

$$p^{(1)}=\tilde{B}_2-4\tilde{B}_1R^2,$$

$$\tilde{B}_1=-\frac{(p_{\text{ou}}-p_{\text{in}})}{2p_{\text{in}}(R_{\text{ou}}^2-R_{\text{in}}^2)},\quad \tilde{B}_2=-\frac{p_{\text{ou}}R_{\text{in}}^2-p_{\text{in}}R_{\text{ou}}^2}{p_{\text{in}}(R_{\text{ou}}^2-R_{\text{in}}^2)}. \tag{3.45}$$

The displacement field and stresses for the second-order problem are given by Eqs. (3.22) and (3.23), respectively, with constants B_1 and B_3 replaced by $\tilde{B}_1$ and $\tilde{B}_3$, respectively. The pressure field is now given by

$$p^{(2)}(R)=\tilde{B}_4-4\tilde{B}_3R^2+12\tilde{B}_1^2\ln R,$$

$$\tilde{B}_3=\frac{3(p_{\text{ou}}-p_{\text{in}})^2\ln(R_{\text{ou}}/R_{\text{in}})}{2p_{\text{in}}^2(R_{\text{ou}}^2-R_{\text{in}}^2)^3},$$

$$\tilde{B}_4=\frac{(p_{\text{ou}}-p_{\text{in}})^2}{4p_{\text{in}}^2(R_{\text{ou}}^2-R_{\text{in}}^2)^3}\{\alpha_0(R_{\text{in}}^2-R_{\text{ou}}^2)+4[(3\ln(R_{\text{ou}})-1)R_{\text{ou}}^2-(3\ln(R_{\text{ou}})-1)R_{\text{in}}^2]\}. \tag{3.46}$$

Hence the complete solution of the problem for $n=4$ is

$$u(R)=-\frac{(p_{\text{ou}}-p_{\text{in}})}{2(R_{\text{ou}}^2-R_{\text{in}}^2)}\frac{1}{R}+\frac{(p_{\text{ou}}-p_{\text{in}})^2}{8(R_{\text{ou}}^2-R_{\text{in}}^2)^2}\left[\frac{12\ln(R_{\text{ou}}/R_{\text{in}})}{R}-\frac{1}{R^3}\right],$$

$$p(R)=-\frac{p_{\text{ou}}R_{\text{in}}^2-p_{\text{in}}R_{\text{ou}}^2}{R_{\text{ou}}^2-R_{\text{in}}^2}+\frac{2(p_{\text{ou}}-p_{\text{in}})}{R_{\text{ou}}^2-R_{\text{in}}^2}R^2+\frac{(p_{\text{ou}}-p_{\text{in}})^2}{(R_{\text{ou}}^2-R_{\text{in}}^2)^3}\left\{\frac{\alpha_0}{4}(R_{\text{ou}}^2-R_{\text{in}}^2)\right.$$
$$\left.-[(3\ln(R_{\text{in}})-1)R_{\text{ou}}^2-(3\ln(R_{\text{ou}})-1)R_{\text{in}}^2]-6\ln(R_{\text{ou}}/R_{\text{in}})R^2+3(R_{\text{ou}}^2-R_{\text{in}}^2)\ln R\right\},$$

$$T_{RR}=\frac{p_{\text{in}}(R^2-R_{\text{ou}}^2)+p_{\text{ou}}(-R^2+R_{\text{in}}^2)}{R_{\text{ou}}^2-R_{\text{in}}^2}+f_{10}(R)+f_{11}R^{-2}+f_{12}R^2,$$

$$T_{\Theta\Theta}=\frac{p_{\text{in}}(3R^2-R_{\text{ou}}^2)+p_{\text{ou}}(-3R^2+R_{\text{in}}^2)}{R_{\text{ou}}^2-R_{\text{in}}^2}+f_{20}(R)+f_{21}R^{-2}+f_{22}R^2, \tag{3.47}$$

where

$$f_{10}(R) = -\frac{(p_{\rm ou} - p_{\rm in})^2[-(1 + 6\ln(R_{\rm in}/R)R_{\rm ou}^2 + (1 + 6\ln(R_{\rm ou}/R))R_{\rm in}^2}{2(R_{\rm ou}^2 - R_{\rm in}^2)^3},$$

$$f_{11} = -\frac{p_{\rm in}^2 R_{\rm ou}^2 + p_{\rm ou}^2 R_{\rm in}^2 - p_{\rm ou} p_{\rm in}(R_{\rm ou}^2 + R_{\rm in}^2)}{2(R_{\rm ou}^2 - R_{\rm in}^2)^2}, \quad f_{12} = \frac{3\ln(R_{\rm ou}/R_{\rm in})(p_{\rm ou} - p_{\rm in})^2}{(R_{\rm ou}^2 - R_{\rm in}^2)^3},$$

$$f_{20}(R) = -\frac{(p_{\rm ou} - p_{\rm in})^2[(5 + 6\ln(R/R_{\rm in}))R_{\rm ou}^2 - (5 + 6\ln(R/R_{\rm ou})R_{\rm in}^2]}{2(R_{\rm ou}^2 - R_{\rm in}^2)^3},$$

$$f_{21} = \frac{p_{\rm in}^2 R_{\rm ou}^2 + p_{\rm ou}^2 R_{\rm in}^2 - p_{\rm ou} p_{\rm in}(R_{\rm ou}^2 + R_{\rm in}^2)}{2(R_{\rm ou}^2 - R_{\rm in}^2)^2}, \quad f_{22} = \frac{9\ln(R_{\rm ou}/R_{\rm in})(p_{\rm ou} - p_{\rm in})^2}{(R_{\rm ou}^2 - R_{\rm in}^2)^3}. \tag{3.48}$$

For $n = 4$, the through-the-thickness distributions of the radial and the hoop stresses in a linear elastic FG cylinder have not been plotted since they are similar to those for $n = \frac{3}{2}$.

4. Analytical solution for the affine variation of elastic moduli

For the dependence of μ and α upon the non-dimensional radial coordinate R given by

$$\mu(R) = \mu_0(1 + mR), \quad \alpha(R) = \alpha_0(1 + mR), \tag{4.1}$$

and stresses normalized by μ_0, the complete solution of the second-order problem is given below:

$$u = \frac{(-p_{\rm ou} + p_{\rm in})R_{\rm ou}^2 R_{\rm in}^2}{2R(R_{\rm ou} - R_{\rm in})(R_{\rm in} + R_{\rm ou}(1 + 2mR_{\rm in}))} - \frac{(-p_{\rm ou} + p_{\rm in})^2 R_{\rm ou}^4 R_{\rm in}^4}{8R^3(R_{\rm ou} - R_{\rm in})^2(R_{\rm in} + R_{\rm ou}(1 + 2mR_{\rm in}))^2}$$
$$+ \frac{(p_{\rm ou} - p_{\rm in})^2 R_{\rm ou}^2 R_{\rm in}^2[3R_{\rm in}^3 + R_{\rm ou}(3 + 4nR_{\rm in})(R_{\rm ou}^2 + R_{\rm ou}R_{\rm in} + R_{\rm in}^2)]}{8R(R_{\rm ou} - R_{\rm in})^2(R_{\rm in} + R_{\rm ou}(1 + 2mR_{\rm in}))^3}. \tag{4.2}$$

$$p = \frac{1}{\beta_D}[m(-p_{\rm ou} + p_{\rm in})R_{\rm ou}^2 R_{\rm in}^2/R + (-p_{\rm in}R_{\rm in}^2 + p_{\rm ou}R_{\rm ou}^2) + 2mR_{\rm in}R_{\rm ou}(p_{\rm ou}R_{\rm ou} - p_{\rm in}R_{\rm in})]$$
$$+ \frac{(p_{\rm ou} - p_{\rm in})^2 R_{\rm ou}^2 R_{\rm in}^2}{\beta_1}\left[\frac{m\alpha_0 R_{\rm ou}^2 R_{\rm in}^2}{R^3} + \frac{m(3R_{\rm in}^2 + R_{\rm ou}(3 + 4mR_{\rm in})(R_{\rm ou}^2 + R_{\rm ou}R_{\rm in} + R_{\rm in}^2)}{R}\right.$$
$$\left. -(3R_{\rm in}(1 + 2mR_{\rm in}) + 2mR_{\rm ou}^2(3 + 4nR_{\rm in}) + R_{\rm ou}^2(3 + 10mR_{\rm in} + 8m^2R_{\rm in}^2)) + \frac{R_{\rm ou}^2}{R^4}(1 + \alpha_0)\right], \tag{4.3}$$

$$\beta_1 = 4(R_{\rm ou} - R_{\rm in})^2(R_{\rm in} + R_{\rm ou}(1 + 2m))^3,$$

$$\beta_D = (R_{\rm ou} - R_{\rm in})(R_{\rm in} + R_{\rm ou}(1 + 2mR_{\rm in})),$$

$$T_{RR} = \frac{1}{\beta_D R^2}[p_{\rm in}(R - R_{\rm ou})(R + (1 + 2mR)R_{\rm ou})R_{\rm in}^2 - p_{\rm ou}R_{\rm ou}^2(R - R_{\rm in})(R + (1 + 2mR)R_{\rm in})]$$
$$+ \frac{1}{\beta_1}\left[g_{10} + \frac{2g_{10}}{R} + \frac{g_{11}}{R^2} + \frac{g_{12}}{R^4}\right],$$

$$T_{\Theta\Theta} = \frac{1}{\beta_D R^2}[p_{\rm in}(R^2 + 2mR^2R_{\rm ou} + R_{\rm ou}^2)R_{\rm in}^2 - p_{\rm ou}R_{\rm ou}^2(R^2 + 2mR^2R_{\rm in} + R_{\rm in}^2)] + \frac{1}{\beta_1}\left[g_{20} + \frac{g_{22}}{R^2} + \frac{g_{23}}{R^4}\right], \tag{4.4}$$

where

$$g_{10} = (p_{\rm ou} - p_{\rm in})^2 R_{\rm ou}^2 R_{\rm in}^2[3R_{\rm in}(1 + 2mR_{\rm in}) + 2mR_{\rm ou}^2(3 + 4mR_{\rm in}) + R_{\rm ou}(3 + 10mR_{\rm in} + 8m^2R_{\rm in}^2)],$$

$$g_{11} = m(p_{\rm ou} - p_{\rm in})^2 R_{\rm ou}^2 R_{\rm in}^2[3R_{\rm in}^3 + R_{\rm ou}^3(3 + 4mR_{\rm in}) + R_{\rm ou}^2R_{\rm in}(3 + 4nR_{\rm in}) + R_{\rm ou}R_{\rm in}^2(3 + 4mR_{\rm in})],$$

$$g_{12} = R_{\rm ou}^2 R_{\rm in}^2(-4p_{\rm ou}p_{\rm in}(R_{\rm ou} + R_{\rm in})(-R_{\rm in}^2 + R_{\rm ou}^2(-1 + 2m^2R_{\rm in}^2)) + p_{\rm ou}^2(-R_{\rm ou}^2R_{\rm in} - 3R_{\rm in}^3 - R_{\rm ou}R_{\rm in}^2(3 + 4mR_{\rm in})$$
$$+ R_{\rm ou}^3(-1 + 4mR_{\rm in} + 8m^2R_{\rm in}^2)) - p_{\rm in}^2(R_{\rm in}^3 + R_{\rm ou}^3(3 + 4mR_{\rm in}) + R_{\rm ou}(R_{\rm in}^3 - 4mR_{\rm in}^3) + R_{\rm ou}^2(3R_{\rm in} - 8mR_{\rm in}^2)),$$

$$g_{20} = (p_{\rm ou} - p_{\rm in})^2 R_{\rm ou}^2 R_{\rm in}^2[3R_{\rm in}(1 + 2mR_{\rm in}) + 2mR_{\rm ou}^2(3 + 4mR_{\rm in}) + R_{\rm ou}(3 + 10mR_{\rm in} + 8m^2R_{\rm in}^2)],$$

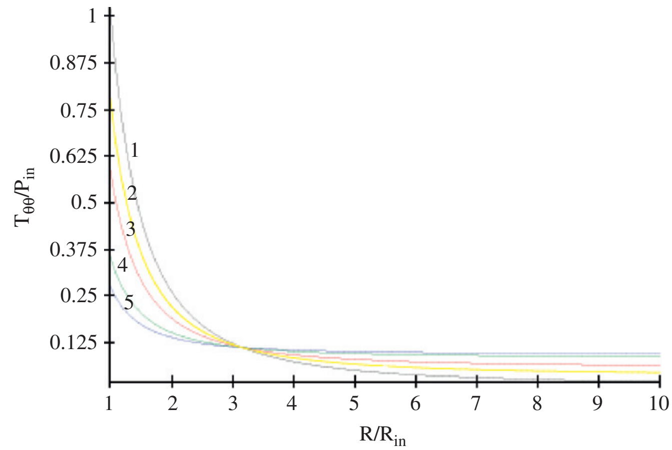

Fig. 4. For $R_{ou}/R_{in} = 10$ and different values of m, variation of the hoop stress through the cylinder thickness. Curves 1 (black line), 2 (yellow line), 3 (red line), 4 (green line), 5 (blue line) are for $m = 0$, 0.2, 0.5, 1.5 and 2.5, respectively.

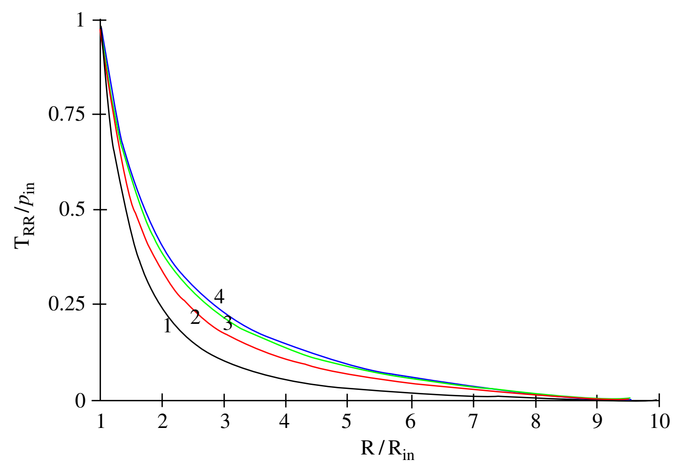

Fig. 5. For $R_{ou}/R_{in} = 10$ and different values of m, variation of the magnitude of the radial stress through the cylinder thickness. Curves 1 (black line), 2 (red line), 3 (green line) and 4 (blue line) are for $m = 0$, 0.5, 1.5 and 2.5, respectively.

$$
\begin{aligned}
g_{22} = {} & R_{ou}^2 R_{in}^2 [p_{ou}^2 R_{in} + 3R_{in}^3 + R_{ou} R_{in}^2 (3 + 4mR_{in}) + R_{ou}^3 (1 - 4mR_{in} - 8m^2 R_{in}^2)] + 4 p_{ou} p_{in} (R_{ou} + R_{in})(-R_{in}^2 \\
& + R_{ou}^2 (-1 + 2m^2 R_{in}^2) + p_{in}^2 (R_{in}^2 + R_{ou}^2 (3 + 4mR_{in}) + R_{ou} R_{in}^2 (1 - 4mR_{in}) + R_{ou}^2 R_{in} (3 - 8mR_{in}^2), \\
g_{23} = {} & 3(p_{ou} - p_{in})^2 R_{ou}^4 R_{in}^4. \qquad (4.5)
\end{aligned}
$$

For $m = 0$ one recovers solutions (3.32) for the homogeneous second-order elastic cylinder.

For a linear elastic FG cylinder loaded only on the inner surface, the radial and the hoop stresses are obtained by setting $p_{ou} = 0$ in the first bracketed term on the right-hand sides of Eqs. (4.4) and ignoring the second bracketed term. For $m \geqslant 0$, T_{RR} is compressive throughout the cylinder thickness and $T_{\Theta\Theta}$ is tensile; $T_{\Theta\Theta}$ is a monotonically decreasing function of R and T_{RR} is a monotonically increasing function of R. For $m_1 \neq m_2$, $T_{\Theta\Theta}(R, m_1) - T_{\Theta\Theta}(R, m_2) = p_{in}(R^2 - R_{in} R_{ou}) f(R, R_{in}, R_{ou}, m_1, m_2)$. Thus at $R = (R_{in} R_{ou})^{1/2}$, the hoop stress is independent of the value assigned to the inhomogeneity parameter m and is the same as that in a homogeneous linear elastic material. In Figs. 4 and 5, variations through the thickness of $T_{\Theta\Theta}/p_{in}$ and T_{RR}/p_{in} are plotted for $R_{in}/R_{ou} = \frac{1}{10}$ and different values of m. It is clear that all curves for $T_{\Theta\Theta}$ intersect at the point $R = \sqrt{10} = 3.16$. However, for all values of m, T_{RR} decreases monotonically from the inner radius to the outer radius.

5. Exponential variation of elastic moduli

The analytical solution involves lengthy expressions, and is thus omitted.

6. Pressurized cylindrical hole in an infinite space

In order to study this problem we assume that $p_{\text{ou}} = 0$ and $R_{\text{ou}} \to \infty$. Thus length like variables are non-dimensionalized with R_{in} rather than with R_{ou}. An example of such a problem is a pressurized cylindrical hole in the ground.

6.1. Power law variation of the elastic moduli

For elastic moduli given by Eq. (2.8), the radial and the hoop stresses for $n \neq 2, 4$ are given by

$$T_{RR} = \begin{cases} -p_{\text{in}}, & n > 2, \\ -p_{\text{in}}\left(\dfrac{R_{\text{in}}}{R}\right)^{2-n} + \dfrac{p_{\text{in}}^2 R_{\text{in}}^{2-2n}(n-2)R^{-2+n}}{2(n-4)}\left[\dfrac{3(n-2)}{2} - \dfrac{(n-3)R_{\text{in}}^2}{R^2}\right], & n < 2, \end{cases}$$

$$T_{\Theta\Theta} = \begin{cases} -p_{\text{in}}, & n > 2, \\ p_{\text{in}}(1-n)\left(\dfrac{R_{\text{in}}}{R}\right)^{2-n} + \dfrac{p_{\text{in}}^2 R_{\text{in}}^{2-2n}(n-2)(n-1)R^{-2+n}}{2(n-4)}\left[\dfrac{3(n-2)}{2} - \dfrac{(n-3)R_{\text{in}}^2}{R^2}\right], & n < 2. \end{cases} \tag{6.1}$$

For $n = 1$, $T_{\Theta\Theta} = 0$. The hoop stress for the first-order problem changes sign as n is increased from a value less than one to a value greater than one.

For $n = 2$ and 4, the stresses are given by

$$T_{RR} = T_{\Theta\Theta} = -p_{\text{in}}. \tag{6.2}$$

6.2. Affine variation of the elastic moduli

For affine variations of the elastic moduli given by Eq. (4.1), the radial and the hoop stresses have the following expressions:

$$T_{RR} = -\frac{(1+2mR)p_{\text{in}}R_{\text{in}}^2}{(1+2mR_{\text{in}})R^2} + \frac{p_{\text{in}}^2 R_{\text{in}}^2}{2(1+2mR_{\text{in}})^2}\frac{1}{R}\left[\frac{(3+4mR_{\text{in}})}{(1+2mR_{\text{in}})}\left(-m - \frac{1}{R}\right) + \frac{3R_{\text{in}}^2}{R^3}\right],$$

$$T_{\Theta\Theta} = \frac{p_{\text{in}}R_{\text{in}}^2}{(1+2mR_{\text{in}})R^2} + \frac{p_{\text{in}}^2 R_{\text{in}}^2}{4(1+2mR_{\text{in}})^2}\frac{1}{R^2}\left[\frac{(3+4mR_{\text{in}})}{(1+2mR_{\text{in}})} - \frac{3R_{\text{in}}^2}{R^2}\right]. \tag{6.3}$$

For a homogeneous cylinder, stresses obtained from either Eq. (6.1) by setting $n = 0$ or from Eq. (6.3) with $m = 0$ equal those given in Eqs. (3.35).

7. Conclusions

We have provided analytical expressions for displacements and stresses induced in a functionally graded cylinder by pressures applied to its inner and outer surfaces. The cylinder material is isotropic and incompressible second-order elastic with all material moduli having similar variation in the radial direction. For a homogeneous linear elastic material, our results reduce to those given in a book on linear elasticity. For a linear through-the-thickness variation of the shear modulus, the hoop stress for the first-order problem is constant throughout the thickness, but not for the second-order problem. Also, the hoop stress for the first-order problem vanishes when pressures on the inner and the outer surfaces are inversely proportional to their radii. For an affine variation of the shear modulus, the hoop stress at $R = \sqrt{R_{\text{in}}R_{\text{ou}}}$ in an internally loaded cylinder is independent of the inhomogeneity parameter, and equals that in a homogeneous cylinder. Here R, R_{in} and R_{ou} equal, respectively, the radial coordinate of a point, the inner radius of the cylinder, and the outer radius of the cylinder. The value of the second-order elastic modulus can be ascertained by measuring the total axial force at the end faces of the cylinder.

Acknowledgments

This work was partially supported by the Office of Naval Research Grant N00014-06-1-0567 to Virginia Polytechnic Institute and State University with Dr. Y.D.S. Rajapakse as the program manager. Views expressed in the paper are those of the authors and neither of the funding agency nor of their institution.

References

[1] S.G. Lekhnitskii, Theory of Elasticity of an Anisotropic Body, Mir Publishers, Moscow, 1981.

[2] R.C. Batra, Finite plane strain deformations of rubberlike materials, Int. J. Numer. Methods Eng. 15 (1980) 145–160.
[3] M.B. Bever, P.E. Duwez, Gradients in composite materials, Mater. Sci. Eng. 10 (1972) 1–8.
[4] M. Shen, M.B. Bever, Gradients in polymeric materials, J. Mater. Sci. 7 (1972) 741–746.
[5] M. Yamanouchi, M. Koizumi, T. Hirai, I. Shiota, in: Proceedings of the International Symposium on Functionally Graded Materials, Sendai, Japan, 1990.
[6] S.S. Vel, R.C. Batra, Exact solutions for thermoelastic deformations of functionally graded thick rectangular plates, AIAA J. 40 (7) (2002) 1421–1433.
[7] S.S. Vel, R.C. Batra, Three-dimensional exact solution for the vibration of functionally graded rectangular plates, J. Sound Vib. 272 (2004) 703–730.
[8] C.O. Horgan, A.M. Chan, The pressurized hollow cylinder or disk problem for functionally graded isotropic linear elastic materials, J. Elasticity 55 (1999) 43–59.
[9] C.O. Horgan, A.M. Chan, Torsion of functionally graded isotropic linear elastic bars, J. Elasticity 52 (1999) 181–199.
[10] C.O. Horgan, A.M. Chan, The stress response of functionally graded isotropic linear elastic rotating disks, J. Elasticity 55 (1999) 219–230.
[11] R.C. Batra, Torsion of a functionally graded cylinder, AIAA J. 44 (6) (2006) 1363–1365.
[12] E. Pan, F. Han, Exact solution for functionally graded and layered magneto-electro-elastic plates, Int. J. Eng. Sci. 43 (2005) 321–339.
[13] Z.-Q. Cheng, R.C. Batra, Three-dimensional thermoelastic deformations of a functionally graded elliptic plate, Composites B 31 (2000) 97–106.
[14] Z.Q. Cheng, R.C. Batra, Exact correspondence between eigenvalues of membranes and functionally graded simply supported polygonal plates, J. Sound Vibration 229 (2000) 879–895.
[15] Z.-Q. Cheng, R.C. Batra, Deflection relationships between the homogeneous plate theory and different functionally graded plate theories, Arch. Mech. 52 (2000) 143–158.
[16] Z.H. Jin, R.C. Batra, Some basic fracture mechanics concepts in functionally graded materials, J. Mech. Phys. Solids 44 (1996) 1221–1235.
[17] Z.H. Jin, R.C. Batra, Stress intensity relaxation at the tip of an edge crack in a functionally graded material subjected to a thermal shock, J. Therm. Stresses 19 (1996) 317–339.
[18] R.C. Batra, J. Jin, Natural frequencies of a functionally graded anisotropic rectangular plate, J. Sound Vib. 282 (2005) 509–516.
[19] S. Vidoli, R.C. Batra, Coupled extensional and torsional deformations of a piezoelectric cylinder, Smart Mater. Struct. 10 (2001) 300–304.
[20] R.C. Batra, B.M. Love, Adiabatic shear bands in functionally graded materials, J. Therm. Stresses 27 (2004) 1101–1123.
[21] A. Signorini, Sulle deformazioni termoelastiche finite, Proceedings of the International Congress on Applied Mechanics, vol. 2, 1930, pp. 80–89.
[22] C.A. Truesdell, W. Noll, The Nonlinear Field Theory of Mechanics, Handbuch der Physik, III/3, Springer, Berlin, 1965.
[23] G.L. Iaccarino, A. Marasco, A. Romano, Signorini's method for live loads and second-order effects, Int. J. Eng. Sci. 44 (2006) 312–324.
[24] F. dell'Isola, G.C. Ruta, R.C. Batra, A second-order solution of Saint-Venant's problem for an elastic pretwisted bar using Signorini's perturbation method, J. Elasticity 49 (1998) 113–127.
[25] F. dell'Isola, G.C. Ruta, R.C. Batra, Generalized Poynting effects in predeformed prismatic bars, J. Elasticity 50 (1998) 181–196.
[26] R.C. Batra, F. Dell'Isola, G.C. Ruta, Second-order solution of Saint-Venant's problem for an elastic bar predeformed in flexure, Int. J. Nonlinear Mech. 40 (2005) 411–422.
[27] J.Q. Tarn, Exact solutions for functionally graded anisotropic cylinders subjected to thermal and mechanical loads, Int. J. Solids Struct. 38 (46–47) (2001) 8189–8206.
[28] J.Q. Tarn, H.H. Chang, Torsion of cylindrical orthotropic elastic circular bars with radial inhomogeneity: some exact solutions and end effects, Int. J. Solids Struct. 45 (1) (2008) 303–319.
[29] V.I. Alshits, H.O.K. Kirchner, Cylindrically anisotropic, radially inhomogeneous elastic materials, Proc. Roy. Soc. Lond. Ser. A — Math., Phys. Eng. Sci. 457 (2007) (2001) 671–693.
[30] A. Oral, G. Anlas, Effects of radially varying moduli on stress distribution of nonhomogeneous anisotropic cylindrical bodies, Int. J. Solids Struct. 42 (20) (2005) 5568–5588.
[31] K.M. Liew, S. Kitiporncai, X.Z. Zhang, C.W. Lim, Analysis of the thermal stress behavior of functionally graded hollow circular cylinders, Int. J. Solids Struct. 44 (10) (2003) 2355–2380.
[32] M. Jabbari, S. Sohrabpour, M.R. Eslami, General solution for mechanical and thermal stresses in a functionally graded hollow cylinder due to nonaxisymmetric steady state loads, J. Appl. Mech. — T ASME 70 (1) (2003) 111–118.
[33] Y. Obata, N. Noda, Steady thermal stresses in a hollow circular cylinder subjected and a hollow sphere of a functionally graded material, J. Thermal Stresses 17 (3) (1994) 471–488.
[34] K.S. Kim, N. Noda, Green's function approach to unsteady thermal stresses in an infinite hollow cylinder of functionally graded material, Acta Mech. 156 (3–4) (2002) 145–161.
[35] E. Pan, A.K. Roy, A simple plane-strain solution for functionally graded multilayered isotropic cylinders, Struct. Eng. Mech. 24 (6) (2006) 727–740.
[36] Z.S. Shao, G.W. Ma, Thermo-mechanical stresses in functionally graded circular hollow cylinder with linearly increasing boundary temperature, Composite Struct. 83 (1) (2008) 259–265.
[37] S.-R. Li, R.C. Batra, buckling of axially compressed cylindrical shells with functionally graded middle layer, Thin-Walled Struct. 44 (2006) 1039–1047.
[38] R.C. Batra, Optimal design of functionally graded incompressible linear elastic cylinders and spheres, AIAA J., in press, doi: 10.2514/1.34937.
[39] B.M. Love, R.C. Batra, Determination of effective thermomechanical parameters of a mixture of two thermoviscoplastic constituents, Int. J. Plasticity 22 (2006) 1026–1061.
[40] R.C. Batra, Elements of Continuum Mechanics, AIAA Press, Vienna, 2005.