

FINITE PLANE STRAIN DEFORMATIONS OF NONLINEAR VISCOELASTIC RUBBER-COVERED ROLLS

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SUMMARY

Finite deformations of a nonlinear viscoelastic rubber layer bonded to a uniformly rotating rigid cylinder and indented by another rigid cylinder are studied by the finite element method. The problem formulation includes both material and geometric nonlinearities. The effect of the change in the thickness of the layer, its speed of rotation and its material properties on the pressure distribution at the contact surface, and other aspects of the deformation of the layer, are studied.

INTRODUCTION

Rubber-covered roll covers similar to the set-up shown in Figure 1 are used in the paper-mill industry and the textile industry. In these and other applications of the roll covers, a relatively thin rubberlike layer bonded to a cylindrical core made of a considerably harder material is indented by another cylinder also made of a hard material. Typically, the indenter and the core are made of steel or granite. The maximum principal strain induced in the rubberlike layer in these applications seems to exceed 20 per cent. Also, depending upon how the rubber was cured and other factors (such as the environmental temperature), the rubberlike layer may behave viscoelastically. Earlier, Batra² studied the deformations of these roll covers wherein the rubber has been modelled as a nonlinear elastic material. Herein we assume that the rubberlike layer can be modelled as a nonlinear viscoelastic material and study its finite deformations. We assume that sufficient time has elapsed since the start-up of the operation for the transient effects to die down and consequently we study the steady-state problem wherein the rubberlike layer has a constant angular velocity of Ω rad/sec.

We briefly review some of the previous work done on this and other related problems. For the case of infinitesimal deformations of the rubberlike layer presumed to be made of a Hookean material, Hahn and Levinson² used the Airy stress function to solve the problem. They also showed that the effect of Coulomb friction at the contact surface on the deformations of the layer was negligible for moderate values of the coefficient of friction. Soong and Li³ used the same Airy stress function, but used the point-matching method rather than the Gram-Schmidt orthonormalization method used by Hahn and Levinson. In a later paper,⁴ Soong and Li considered the effect of a thin sheet of paper being pressed between two rubber-covered rollers. The deformations of the roll covers were assumed to be infinitesimal.

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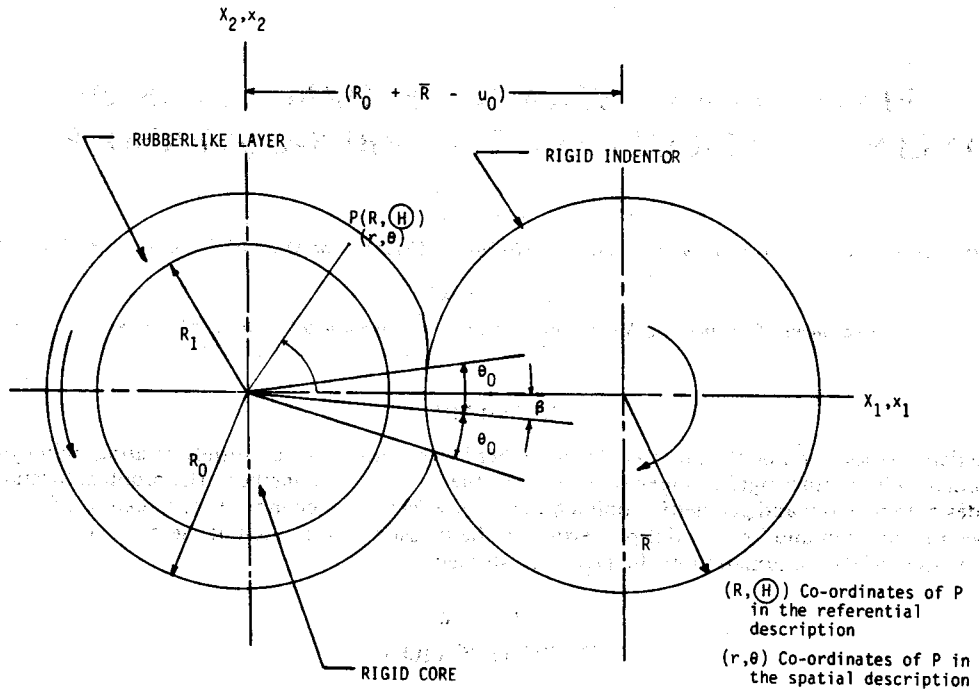


Figure 1. System to be studied

Contact problems in which one of the bodies is made of a homogeneous linear viscoelastic material and the other is taken to be rigid have been studied by Hunter,⁵ Morland^{6,7} and Harvey.⁸ These studies have entailed the contact of a viscoelastic cylinder with a rigid plane or that of a rigid cylinder with a viscoelastic half-space. Various transform methods have been used to reduce the problem to that for an elastic case. A contact problem involving the cold rolling of a viscoelastic sheet was solved by the finite element method by Lynch.⁹ In this problem, the boundary surface where surface tractions are prescribed is not a material surface. Lynch studied the steady-state rolling process and consequently was able to express the dependence on the past history of the strain rate in the stress-strain law on the spatial variation of the strain. Thus, the resulting stiffness matrix had a bandwidth considerably larger than that required to solve an elastic problem. Batra *et al.*¹⁰ used a similar technique and solved the roll cover problem when the rubberlike layer is assumed to be made of a thermorheologically simple material. Both Lynch and Batra *et al.* accounted for the incompressibility of rubber by taking the bulk modulus to be much larger than the shear modulus. Subsequently, Batra¹¹ used an exact formulation for incompressible materials and solved for the finite deformations of nonlinear rubber-covered rolls.

Experimental work involving varying thickness of the rubberlike layer and different combinations of diameters of mating cylinders has been performed by Spengos.¹² We note that in his experimental set up, the ratio of the diameter of roll cover to the length of cylinders is of the order of one. Therefore, in order to compare analytical results with his experimental findings one will need to solve a three-dimensional problem. We also note that the only material property listed in Spengos' work is the durometer hardness.

In this paper we study finite plane strain deformations of the roll covers, assuming that these are made of a nonlinear viscoelastic material. The constitutive relation used is that proposed

recently by Christensen.¹³ We use the formulation valid for incompressible materials and regard pressure in each element as an unknown quantity. Using 4-noded isoparametric elements with 2×2 Gaussian integration rule, we solve for unknown nodal displacements and the discontinuous pressure field. The developed computer code has been tested for accuracy by solving a nonlinear elastic problem solved earlier¹¹ by using a different computer code and also by solving a linear viscoelastic problem studied earlier by Batra.¹⁴ In each case the solution obtained by using the present code compared favourably with that obtained earlier. The parameters whose effect on the deformations of roll covers have been studied in this work include the thickness of the roll cover, the speed of rotation, material moduli and the radius of the indenter. When studying these different cases, the depth of indentation has been kept constant.

FORMULATION OF THE PROBLEM

We use a fixed rectangular Cartesian co-ordinate system with origin at the centre of the roll cover to describe the deformations of the rubberlike layer. We denote the position of a material particle in the reference configuration by X_α and the position of the same material particle in the current configuration at time t by x_i . Thus, $x_i = x_i(X_\alpha, t)$ describes the deformation which is governed by the following equations:

$$\dot{x}_{i,i} = 0 \quad \dot{x}_{i,t} \equiv \frac{\partial \dot{x}_i}{\partial x_i} \quad \rho \ddot{x}_i = \sigma_{ij,j} \quad (1)$$

Equation (1)₁ is the continuity equation for incompressible materials and equation (1)₂ describes the balance of linear momentum. Here, ρ is the present mass density which, for the material being studied, equals the mass density in the reference configuration, a superimposed dot implies material time derivative, σ_{ij} is the Cauchy stress tensor, and a repeated index implies summation over the range of the index. Equations (1) are to be supplemented in general by initial conditions, boundary conditions and a constitutive relation. Before we state these, we give below the assumptions made to make the problem tractable.

We assume that steady state has been reached and the effect of all dynamic forces such as the inertia forces and centrifugal forces on the deformations of the rubberlike layer is negligible. This assumption seems reasonable since the mass density of rubber is quite low, being close to that of water. Also, we will consider only those situations for which the length of the roll covers is very large compared to their diameters. Thus, plane strain state of deformation is assumed to prevail. Under these conditions, $x_3 = X_3$ and equation (1)₃ for $i = 3$ is identically satisfied. Thus, we need to solve

$$\dot{x}_{i,i} = 0 \quad \sigma_{ij,j} = 0 \quad (i = 1, 2) \quad (2)$$

under the following boundary conditions. At the inner surface

$$u_i = x_i - X_\alpha \delta_{i\alpha} = 0, \quad (3)$$

and at the outer surface

$$\begin{aligned} e_i \sigma_{ij} n_j &= 0 \\ n_i \sigma_{ij} n_j &= 0 \quad |\theta - \beta| > \theta_0 \quad \theta \equiv \arctan(x_2/x_1), \\ n_i \sigma_{ij} n_j &= f(\theta) \quad |\theta - \beta| < \theta_0 \\ f(\theta) &\rightarrow 0 \quad \text{as } |\theta - \beta| \rightarrow \theta_0 \end{aligned} \quad (4)$$

In these equations, n_i is a unit outward normal to the boundary and e_i is a unit tangent vector to the boundary in the present configuration. The boundary condition (3) implies that there is perfect bonding at the inner surface between the rubberlike layer and the supporting cylinder. This seems to be supported by experiments, since an examination of failed roll covers indicates that there is always some rubberlike material left on the cylinder. The boundary conditions (4) indicate that the part of the outer surface of the rubberlike layer not in contact with the indenter is traction free. Also, on the part of the outer surface of the roll cover that is in contact with the indenter, there is no traction in the tangential direction and the normal force between the two is given by the function $f(\theta)$. The corner condition (4)₄ implies that a contact problem rather than a punch problem is being solved. What is left to complete the formulation of the problem is the constitutive relation for the rubberlike material. We take this to be the one proposed by Christensen.¹³ That is

$$\sigma_{ij}(x(X, t), t) = -p(x, t)\delta_{ij} + F_{i\alpha}F_{j\beta} \left[g_0\delta_{\alpha\beta} + \int_{-\infty}^t g_1(t-\tau) \frac{\partial E_{\alpha\beta}(X, \tau)}{\partial \tau} d\tau \right]. \quad (5)$$

Here, p is the hydrostatic pressure not determined by the deformation, g_0 and g_1 are material moduli, g_1 describes how the present stress at the material point X depends upon its past deformation history, $g_1(t)$ decays to zero as time t goes to infinity, $\delta_{\alpha\beta}$ is the Kronecker delta and the right Cauchy–Green tensor $C_{\alpha\beta}$ and the Green–Lagrange strain tensor $E_{\alpha\beta}$ are related to the deformation gradient $F_{i\alpha}$ by

$$C_{\alpha\beta} = F_{i\alpha}F_{i\beta} \quad 2E_{\alpha\beta} = C_{\alpha\beta} - \delta_{\alpha\beta} \quad F_{i\alpha} = x_{i,\alpha} \equiv \partial x_i / \partial X_\alpha. \quad (6)$$

In the present work we have taken the material modulus $g_1(t)$ to have the form

$$g_1(t) = g_1(0) \exp(-t/\tau_r) \quad (7)$$

where τ_r is the relaxation time of the material. It is possible that the material may have more than one relaxation time. For simplicity, we have assumed the existence of only one relaxation time. Our procedure and the numerical work can easily be modified to accommodate situations where the material response function $g_1(t)$ is represented by a series rather than the one term. Implicit in the constitutive relation (5) is the assumption that the rubberlike layer is made of a homogeneous and isotropic material.

We note that because of the assumption of steady state, the variable of integration τ in equation (5) can be changed to the spatial co-ordinate θ . Also, in order for our assumption of steady state to be valid, the relaxation time τ_r of the material would have to be such that stresses and strains at a material point relax to zero soon after it leaves the region $|\theta - \beta| \leq \theta_0$. If it were not so, the stresses at a material point would very likely keep on building up and a steady state would not be reached. Thus, the solution technique proposed herein will apply to situations wherein the relaxation time of the rubber and the speed of rotation of the roll cover are suitably related.

The problem as formulated above is not well defined since θ_0 , β and $f(\theta)$ are not known *a priori*. These three should have values such that the boundary conditions (4) are satisfied and the deformed surface of the roll cover in the region $|\theta - \beta| \leq \theta_0$ matches with the circular profile of the indenter. In order to solve the problem, we use the following iterative procedure. We assume θ_0 and β and iterate on the pressure profile $f(\theta)$ until the computed deformed surface within the assumed contact width matches with the circular profile of the indenter. We also ensure that the presumed contact width is correct by ensuring that points on the outer deformed surface outside the assumed contact width have not penetrated into the indenter.

The problem as formulated above is too complicated for us to solve analytically. Therefore, we seek an approximate solution of the problem by using the finite element method.

FINITE ELEMENT FORMULATION OF THE PROBLEM

For the numerical work we use the referential description which is sometimes also called the total Lagrangian description of the deformations. Equations equivalent to equations (2) that govern the deformations of the rubberlike layer may also be obtained by using the principle of virtual work. This principle for incompressible bodies (e.g. see Truesdell and Toupin¹⁵ for one such version of the principle) may be stated as $\delta W = 0$ for all virtual displacements δu_i that vanish on the part of the boundary where essential boundary conditions are prescribed. Also, W is given by

$$W = \int S_{\alpha\beta}^d E_{\alpha\beta} dV - \int h_\alpha u_\alpha dA - \int \frac{p}{2} (I_3 - 1) dV \quad (8)$$

where $S_{\alpha\beta}^d$ may be called the deviatoric part of the second Piola-Kirchhoff stress tensor that contributes to virtual work during a virtual deformation of the body compatible with the incompressibility constraint $I_3 (= \det C) = 1$. Furthermore, in equation (8), h_α is the surface traction per unit area in the reference configuration and all integrations are over the region occupied by the body in the reference configuration. The stress tensor $S_{\alpha\beta}$ is related to the Cauchy stress tensor σ_{ij} by the relation

$$S_{\alpha\beta} = I_3^{-1/2} F_{i\alpha} \sigma_{ij} F_{j\beta} = F_{i\alpha} \sigma_{ij} F_{j\beta} \quad (9)$$

where the second equality holds only for incompressible materials, since these materials can undergo isochoric deformations only. The constitutive relation for $S_{\alpha\beta}$ that is compatible with the constitutive law (5) is

$$S_{\alpha\beta} = S_{\alpha\beta}^d - p(C^{-1})_{\alpha\beta} \quad (10)$$

$$S_{\alpha\beta}^d(X, t) = g_0 \delta_{\alpha\beta} + \int_{-\infty}^t g_1(0) e^{-(t-\tau)/\tau_r} \frac{\partial E_{\alpha\beta}(X, \tau)}{\partial \tau} d\tau \quad (11)$$

Note that the Lagrange multiplier $p/2$ in equation (8) and the hydrostatic pressure p in equation (5) play essentially similar roles since both are meant to account for the kinematic constraint $I_3 = 1$ on the deformations of the rubberlike layer.

Setting $\delta W = 0$, noting that variations δu_i and δp are independent and that $\delta I_3 = 2(C^{-1})_{\alpha\beta} \delta E_{\alpha\beta}$, we obtain

$$\int S_{\alpha\beta} \delta E_{\alpha\beta} dV = \int h_\alpha \delta u_\alpha dA \quad (12)$$

$$\int \frac{\delta p}{2} (I_3 - 1) dV = 0 \quad (13)$$

Assume that the entire load $f(\theta)$ is applied in M equal increments and denote the increment Δu_i in displacements u_i caused by the $(N+1)$ st load increment by Δu_i . That is

$$u_i^{N+1} = u_i^N + \Delta u_i \quad E_{\alpha\beta}^{N+1} = E_{\alpha\beta}^N + \Delta E_{\alpha\beta}, \text{ etc.} \quad (14)$$

Since $\delta E_{\alpha\beta} = \delta \Delta E_{\alpha\beta}$, $\delta p = \delta \Delta p$, equations (12) and (13) for the $(N+1)$ st load increment may be written as

$$\int (S_{\alpha\beta}^N + \Delta S_{\alpha\beta}) \delta \Delta E_{\alpha\beta} dV = \int h_\alpha^{N+1} \delta \Delta u_\alpha dA \quad (15)$$

$$\int \frac{\delta \Delta p}{2} (I_3^N + \Delta I_3 - 1) dV = 0 \quad (16)$$

Using the relations

$$\begin{aligned}
 \Delta E_{\alpha\beta} &= \Delta e_{\alpha\beta} + \Delta \eta_{\alpha\beta} \\
 \Delta e_{\alpha\beta} &= (\Delta u_{\alpha,\beta} + \Delta u_{\beta,\alpha} + u_{i,\alpha}^N \Delta u_{i,\beta} + u_{i,\beta}^N \Delta u_{i,\alpha})/2 \\
 \Delta \eta_{\alpha\beta} &= (\Delta u_{i,\alpha} \Delta u_{i,\beta})/2 \\
 \Delta I_3 &= 2I_3^N (C^{N-1})_{\alpha\beta} \Delta E_{\alpha\beta}
 \end{aligned} \tag{17}$$

we rewrite equations (15) and (16) as follows:

$$\int \Delta S_{\alpha\beta} \delta \Delta E_{\alpha\beta} dV + \int S_{\alpha\beta}^N \delta \Delta \eta_{\alpha\beta} dV = \int h_{\alpha}^{N+1} \delta \Delta u_{\alpha} dA - \int S_{\alpha\beta}^N \delta \Delta e_{\alpha\beta} dV \tag{18}$$

$$\int I_3^N (C^{N-1})_{\alpha\beta} \Delta E_{\alpha\beta} \delta \Delta p dV = - \int \frac{\delta \Delta p}{2} (I_3^N - 1) dV \tag{19}$$

In order to evaluate $\Delta S_{\alpha\beta}$, we first recall our assumption that steady state has been reached. Also, as noted earlier, the stresses and strains at a material point decay to zero as it moves far away from the contact zone. Therefore, we assume that $E_{\alpha\beta} = 0$ at a material point if it lies in the region $|\theta - \beta| > L\theta_0$, where L is a suitable integer. Of course, $L\theta_0 < \pi$. Experience with this problem worked out earlier with different constitutive relations for the rubberlike material indicates that $L = 4$ or 5 . Therefore, for any material point X , the lower limit of integration in the constitutive relation (10) may be set equal to the time when it crosses the line $-\theta - \beta = L\theta_0$, i.e. the line AB in Figure 2. For a material point X we start the clock when it crosses the line AB. Hence, the upper limit of integration in equation (11) equals ψ/Ω , where ψ is the present angular displacement in radians of X from the line AB (cf. Figure 2). Let ψ be divided into n equal intervals with $\psi_0 = 0$, $\psi_1 = \psi/n$, $\psi_2 = 2\psi/n$, ..., $\psi_n = \psi$. The Riemann-Stieltjes integral in equation (11) may be written as a sum of a series. Thus, we rewrite equation (11) as

$$S_{\alpha\beta}^d(X, t) = g_0 \delta_{\alpha\beta} + \sum_{m=1}^n g_1 \left(\frac{\psi - \psi_m^*}{\Omega} \right) [E_{\alpha\beta}(\psi_m) - E_{\alpha\beta}(\psi_{m-1})] \tag{20}$$

where $\psi_{m-1} \leq \psi_m^* \leq \psi_m$. To facilitate the interpretation of this equation, let the region $|\theta - \beta| \leq L\theta_0$ be divided into uniform subregions, as in Figure 2. In each row number the elements temporarily (locally) starting from the end AB. For any material point X , according to equation (20), the value of $S_{\alpha\beta}^d$ at X depends upon the strains the material point X experienced when it passed through the corresponding locations in elements 1 through n of the r th row. From equations (10) and (20) we obtain

$$\begin{aligned}
 \Delta S_{\alpha\beta} &= -\Delta p (C^{N-1})_{\alpha\beta} + 2p^N (C^{N-1})_{\alpha\gamma} (C^{N-1})_{\delta\beta} \Delta E_{\gamma\delta} \\
 &+ \sum_{m=1}^n g_1 \left(\frac{\psi - \psi_m^*}{\Omega} \right) [\Delta E_{\alpha\beta}(\psi_m) - \Delta E_{\alpha\beta}(\psi_{m-1})]
 \end{aligned} \tag{21}$$

We now make the assumption that the load increments are sufficiently small so that

$$\begin{aligned}
 \Delta S_{\alpha\beta} \delta \Delta E_{\alpha\beta} &\approx \Delta \hat{S}_{\alpha\beta} \delta \Delta e_{\alpha\beta} \\
 I_3^N (C^{N-1})_{\alpha\beta} \Delta E_{\alpha\beta} &\approx (C^{N-1})_{\alpha\beta} \Delta e_{\alpha\beta}
 \end{aligned} \tag{22}$$

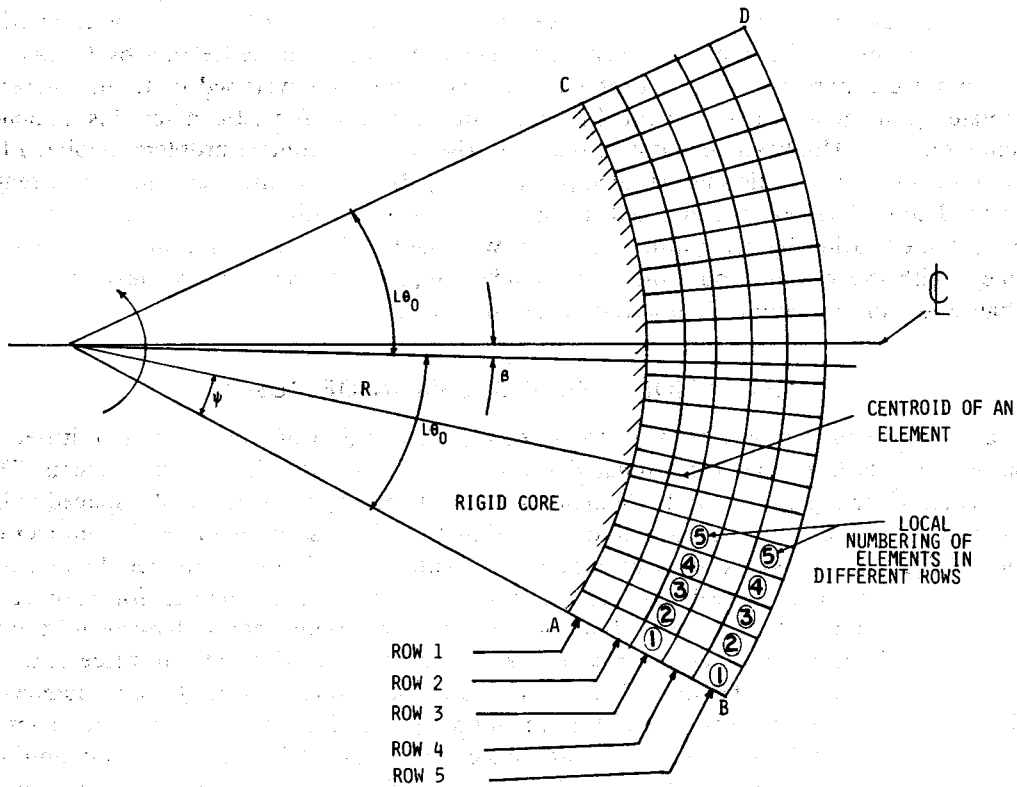


Figure 2. A representative division of the rubberlike layer into finite elements

where $\hat{S}_{\alpha\beta}$ is obtained from $S_{\alpha\beta}$ by substituting $\Delta e_{\alpha\beta}$ for $\Delta E_{\alpha\beta}$ in equation (21). Thus, we rewrite equations (18) and (19) as follows:

$$\int \Delta \hat{S}_{\alpha\beta} \delta \Delta e_{\alpha\beta} dV + \int S_{\alpha\beta}^N \delta \Delta \eta_{\alpha\beta} dV = \int h_{\alpha}^{N+1} \delta \Delta u_{\alpha} dA - \int S_{\alpha\beta}^N \delta \Delta e_{\alpha\beta} dV \quad (23)$$

$$\int (C^{N-1})_{\alpha\beta} \Delta e_{\alpha\beta} \delta \Delta p dV = 0 \quad (24)$$

Note that we have set $I_3^N = 1$ on the right-hand side of equation (19). We use equilibrium iterations,¹⁶ i.e. iterations within a load step to ensure that equations (23) and (24) are satisfied within a prescribed error.

A finite element computer program based on equations (21), (23) and (24), and employing 4-node isoparametric elements with 2×2 Gauss integration rule has been written. In it the hydrostatic pressure p is taken to be constant within each element. We remark that the program developed is a specific purpose one in that it can solve only steady-state viscoelastic problems of the type being studied here. The boundary condition of zero displacements at the interface between the rubberlike layer and the core has been built into the code. This has resulted in fewer equations to be solved. As has been pointed out by Batra *et al.*,¹⁰ the stiffness matrix for this problem is asymmetric and has a much larger bandwidth as compared to that for the corresponding problem in which the rubber is modelled as an elastic material.

The accuracy of the developed finite element code has been established by comparing computed results for the nonlinear elastic problem with those obtained earlier by Batra.¹¹ In order to solve a nonlinear elastic problem with the presently developed code, the material parameter g_1 in equation (5) is set equal to zero. Thus, a nonlinear elastic material is modelled herein by the neo-Hookean constitutive relation. Also, a linear contact problem involving the cold rolling of a viscoelastic sheet studied previously by Batra¹⁴ was analysed with the currently developed code. The results computed for these two test problems compared favourably with those obtained earlier by using different codes. We note that in analysing a linear viscoelastic problem with the present code, the entire load is applied in one step on the undeformed configuration and no equilibrium iterations are performed.

COMPUTATION AND DISCUSSION OF RESULTS

As stated earlier in the paragraph following eqn. (19), a region of roll cover in width equal to approximately five times the estimated contact width is divided into finite elements. This subdivision of the region into finite elements is accomplished by drawing equally spaced radial lines from the centre of the roll cover and by dividing the thickness of the rubber into numerous layers of equal thickness. Numerical experiments with the thickness of rubber divided into three, four and five layers revealed that the division into three layers is adequate for all practical purposes. The values of the total displacement of the nodal point that underwent maximum displacement differed by about 5 per cent when the rubber was divided into three and five layers. Also, because of the limited computer funds available, it was decided somewhat arbitrarily to consider the region of rubber in width equal to 2.5 times the presumed contact width. This will result in less accurate results, since the effect of different boundary conditions on ends AB and CD (see Figure 2) on the deformations of rubber in the vicinity of the contact area may not be insignificant. In the results presented below, the ends AB and CD are taken to be traction free. The equilibrium iterations are stopped when, for each nodal point on the presumed contact width, the increment in the x_1 -component of the displacement caused by the residual (or unbalanced) forces is less than 0.25 per cent of the total x_1 -displacement of that nodal point for the load applied thus far.

To solve a contact problem, contact width $2\theta_0$ and the pressure function $f(\theta)$ are estimated. The boundary value problem is now well defined and is solved numerically. After having found the displacements of nodal points, the deformed shape of the roll cover is determined. A check is now made to ensure that nodal points just outside the presumed contact width have not penetrated into the indenter. If either of the nodes adjacent to but outside of the presumed contact width has gone into the indenter, either the value of $2\theta_0$ is increased or the pressure function $f(\theta)$ is modified and the problem is solved again. However, if the estimated arc of contact is correct, then the values of $f(\theta)$ are modified until the deformed surface of the roll cover within the contact width matches, within the specified tolerance, with the circular profile of the indenter. The tolerance fixed for this comparison is that the distance of each nodal point on the contact surface from the indenter should be less than 1.5 per cent of the indentation u_0 (see Figure 1). The final pressure profile that results in the proper deformed surface is plotted and the value of contact width $2\theta_0$ is read from the graph. Also, the value of β , the angle of asymmetry for the contact width, is determined from this graph. Even though it is feasible to have the contact width start and end at a nodal point and thus not determine θ_0 and β from the plot of $f(\theta)$ vs. θ , the finite element grid needed to achieve this goal would have to be very fine. Since we must have uniform elements in a row, such a fine grid would indeed be very expensive to use.

It should have become clear by now that only an approximate solution of the problem has been obtained. There is no implication that this solution is unique. Also, its closeness to the analytical solution cannot be determined since the latter is unknown and there is no hope of finding one in the near future. With this caution, we now present numerical results for a typical set-up of the roll cover.

The values of various material and geometric parameters selected are:

$$\begin{aligned} R_0 &= 2.39 \text{ in.} & R_1 &= 1.83 \text{ in.} & \bar{R} &= 3.0 \text{ in.} \\ \Omega &= 12.5 \text{ rad/sec} & \tau_r &= 0.002 \text{ sec} \\ g_0 &= 25 \text{ psi} & g_1(0) &= 25 \text{ psi} \end{aligned} \quad (25)$$

Also, to investigate the effect of change in the values of these parameters, one parameter was varied at a time. To shorten the notation in the graphs, we use the identification given in Table I for these problems. We note that problem 2 represents the case when the rubber is taken to be a neo-Hookean (the simplest nonlinear elastic) material. Figure 3 depicts the pressures needed to cause an 0.127 in. indentation in the roll cover. The values of the total load required to deform the roll cover, the values of the contact width $2\theta_0$ in the referential description and the angle β of asymmetry are listed in Table II. The total load P is evaluated from

$$P = \int_{-\theta_0-\beta}^{\theta_0-\beta} f(\theta) d\theta$$

Table I. Problem labelling

Parameter(s) changed	Its new value	Problem No.
None		1
$g_0, g_1(0)$	$g_0 = 50 \text{ psi}, g_1(0) = 0$	2
R_1	1.97 in.	3
Ω	40 rad/sec.	4
$\bar{R}$	2.0 in.	5

Table II. Load and contact width

Problem No.	Load P (lb/in.)	$2\theta_0$ (rad)	β (rad)
1	38.59	0.389	0.0018
2	77.2	0.4389	0.0
3	70.73	0.4109	0.0058
4	41.2	0.3970	0.0076
5	34.3	0.360	0.0039

Note that the pressure distribution on the contact width is asymmetric about the centreline of roll covers. The value of the maximum pressure occurs at a point to the left of the centreline of roll covers. As expected, maximum load is required for the elastic problem. The decrease in the thickness of the roll cover from 0.56 in. to 0.42 in. results in a considerable increase in the total load required to cause the same indentation. For a decrease in the radius of the

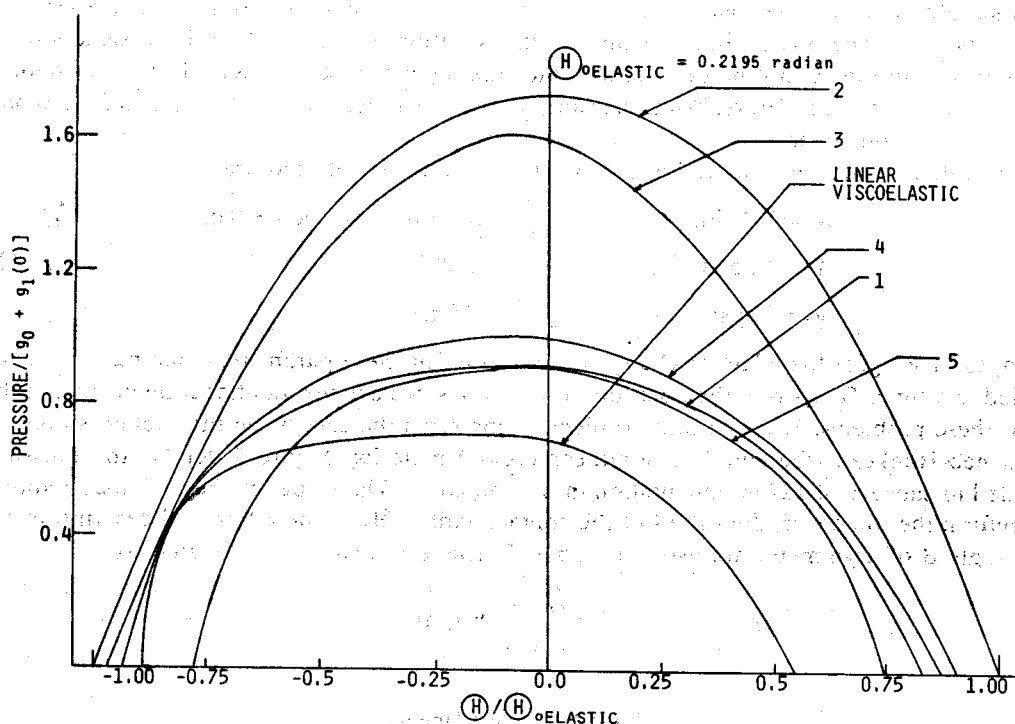


Figure 3. Pressure distribution at the contact surface

indenter from 3 in. to 2 in., both the total load and the contact width decrease. However, an increase in the angular speed of the roll cover from 125 rad/sec to 40 rad/sec is accompanied by a slight increase in the requisite load.

Also, plotted in Figure 3 is the pressure profile for the linear viscoelastic problem. In this case, the entire load is applied in one step on the undeformed outer surface of the roll cover and no equilibrium iterations are performed. The numerical values computed for various parameters are $P = 20.194$ lb, $2(H)_0 = 0.316$ rad, $\beta = 0.1174$ rad. We note that the maximum strains involved are of the order of 40 per cent. It is clear that for this set-up of roll covers and 0.127 in. indentation, the solution for the linear problem differs dramatically from that for the nonlinear problem.

In Figure 4 are shown the undeformed and deformed shapes of the roll cover and the indenter. These plot as ellipses since the scales along horizontal and vertical axes are different. One can determine the contact width from this figure too. Whereas the value of the contact width determined from Figure 3 is in the reference configuration, that calculated from the deformed shape shown in Figure 4 is in the present configuration. Therefore, the two sets of values may not be equal. The change in curvature of the deformed surface on the exit side plays a major role in some applications of the system, as in the paper industry. Depending upon the radius of curvature at point A, the pulp may roll over with the roll, in which case the system will be useless. As the plotted results show, the thickness of the roll cover, the speed of operation and the values of material moduli are some of the factors that determine the change in curvature.

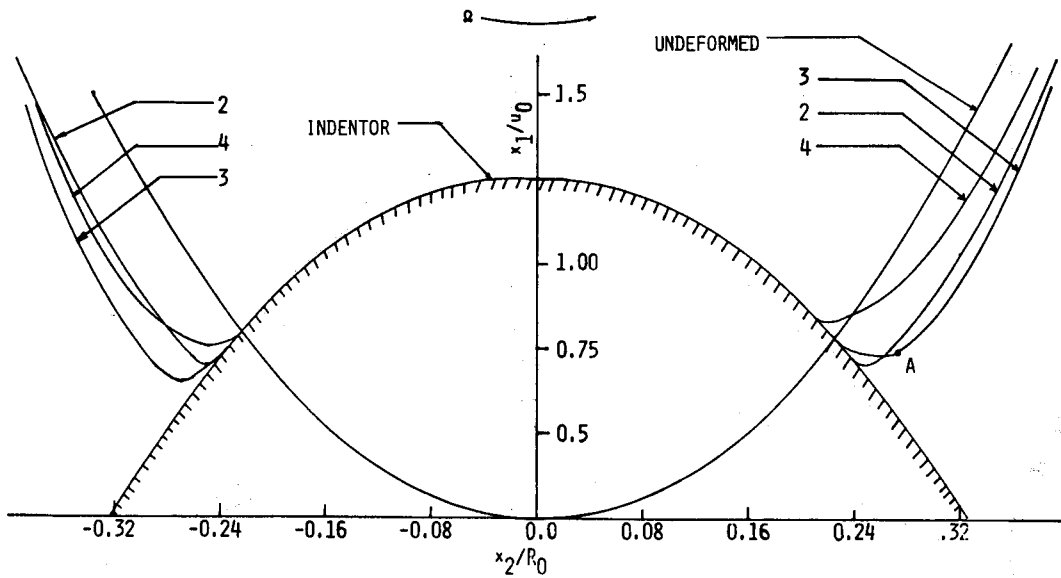


Figure 4. Asymmetric surface deformation of the rubberlike layer

Figures 5–7 show, respectively, the variation of the radial, circumferential and the shear stress with the angular co-ordinate Θ at points near the bond surface. The distance of these points from the bond surface equals 7 per cent of the thickness of the roll cover. All of these stress distributions, except that for the elastic case (problem 2), are asymmetric about the centreline. Even though the peak pressure at the contact surface for the elastic problem was

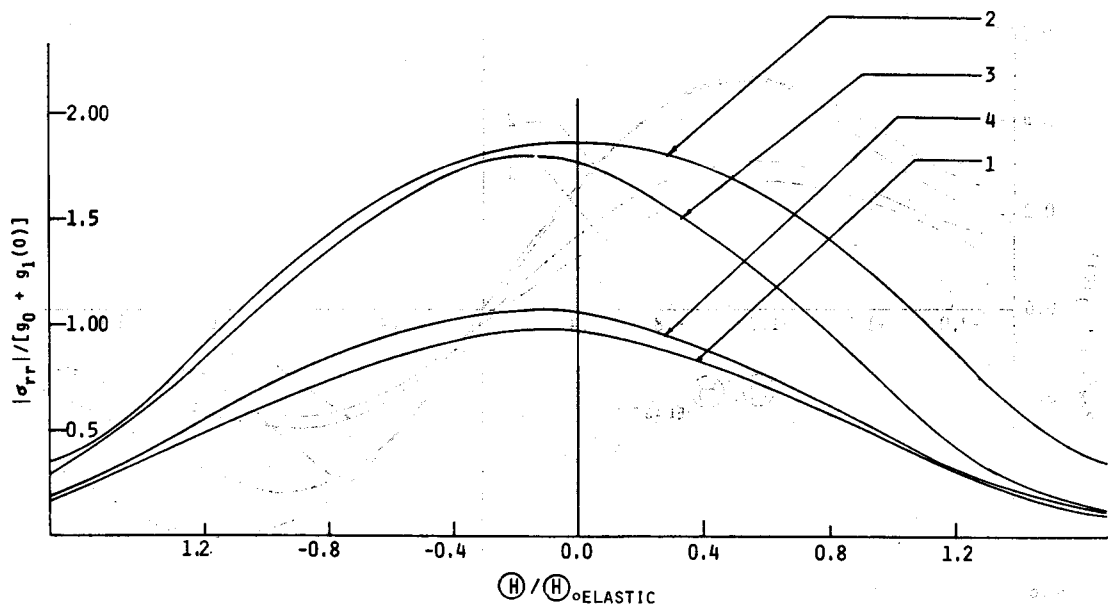


Figure 5. Radial stress distribution near the bond surface

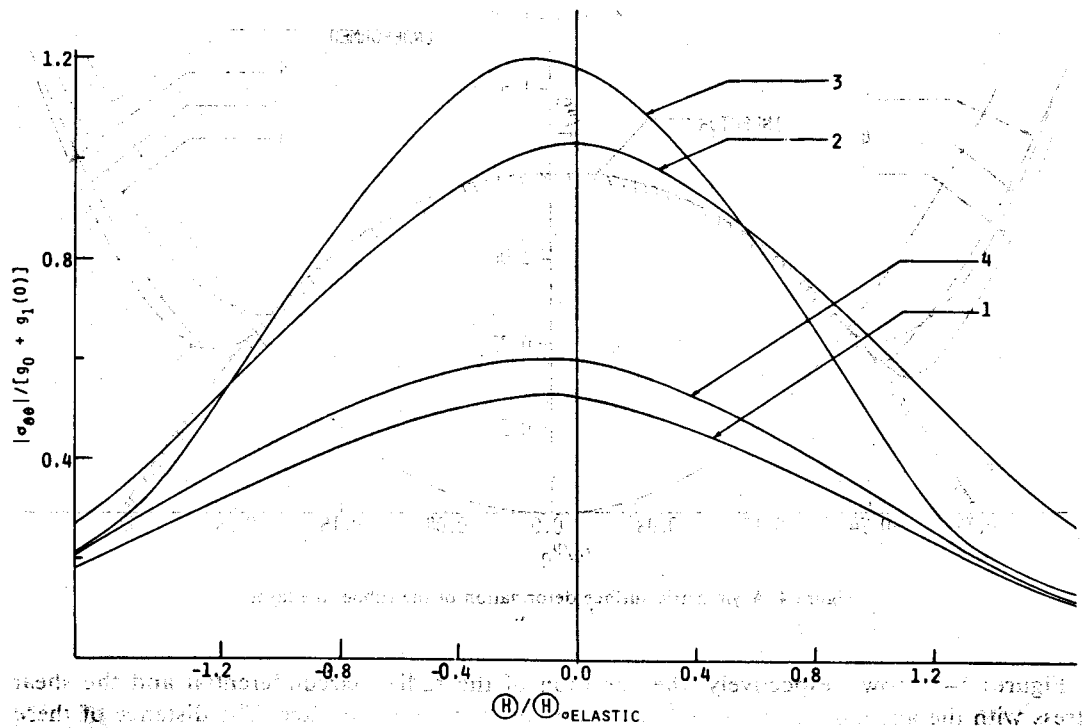


Figure 6. Circumferential stress distribution near the bond surface

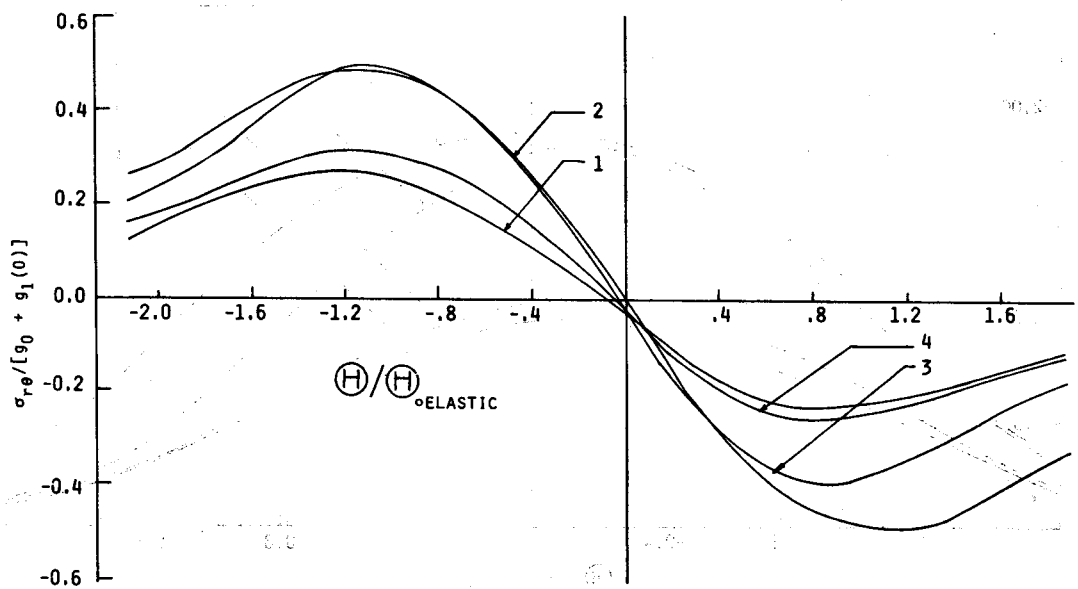


Figure 7. Shear stress distribution near the bond surface

higher than the peak pressure for problem 3, the maximum value of the circumferential stress for problem 3 is more than that for problem 2. The peak values of the radial stress and the circumferential stress occur at points slightly to the left of the centreline of the roll covers. In Figure 7, the curvature of the shear stress curve for viscoelastic problems at points to the left of A and to the right of B is opposite to that for the elastic problem studied. To investigate further the effect of the thickness of the roll cover, we have plotted in Figures 8 and 9 the pressure distribution at the contact surface and the radial stress distribution at points near the bond surface. Recalling that the indentation is kept constant, within the accuracy of our results, the arc of contact increases with the decrease in the thickness and so does the total load required. The point where the maximum radial stress occurs moves further to the left of the centreline as the thickness is decreased.

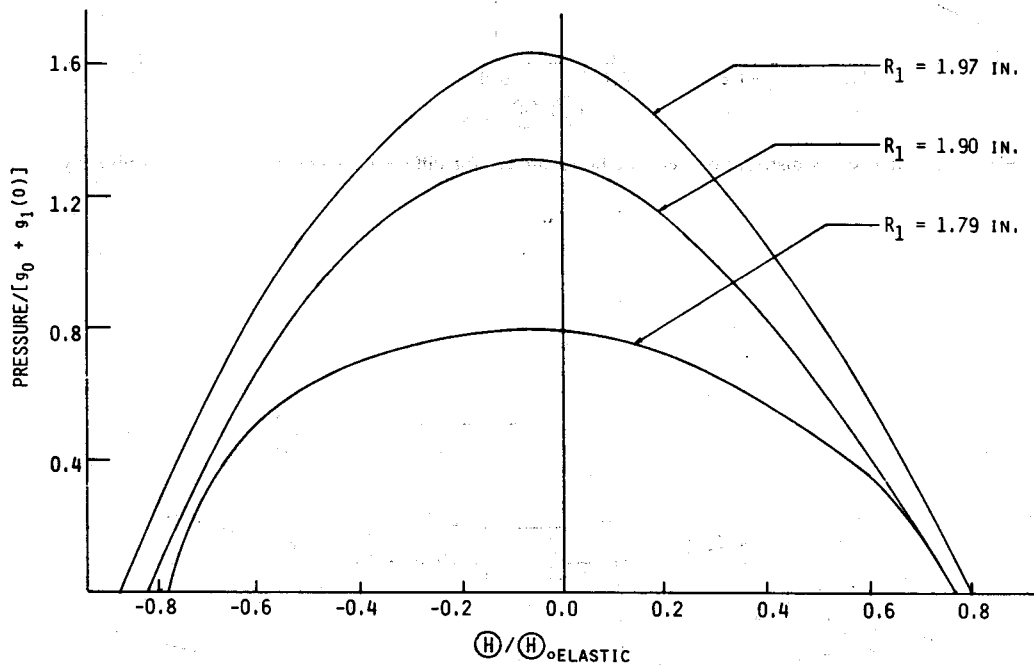


Figure 8. Pressure distribution at the contact surface for different thicknesses of the rubberlike layer

Figure 10 depicts the details of the deformations of the roll cover within the vicinity of the contact width for problem 1. The position of points in the undeformed configuration is denoted by upper case letters; their position in the deformed configuration by the corresponding lower case letters. Since we have used 4-noded isoparametric elements, these have been plotted as quadrilaterals both in the undeformed and deformed configurations. A close look at this figure reveals that there is significant shear deformation of the rubber. For this problem the maximum shear strain that occurs at a point within the rubber is of the order of 40 per cent. The exact magnitude of the maximum shear strain and the point where it occurs depends upon parameters such as the thickness of the roll cover. *For problem 3, the maximum shear strain equals 0.472 and it occurs in the element No. 119 when the roll cover is divided into 3 rows of equal thicknesses with 50 elements in each row. Nearly 16 central elements are within the arc of contact.*

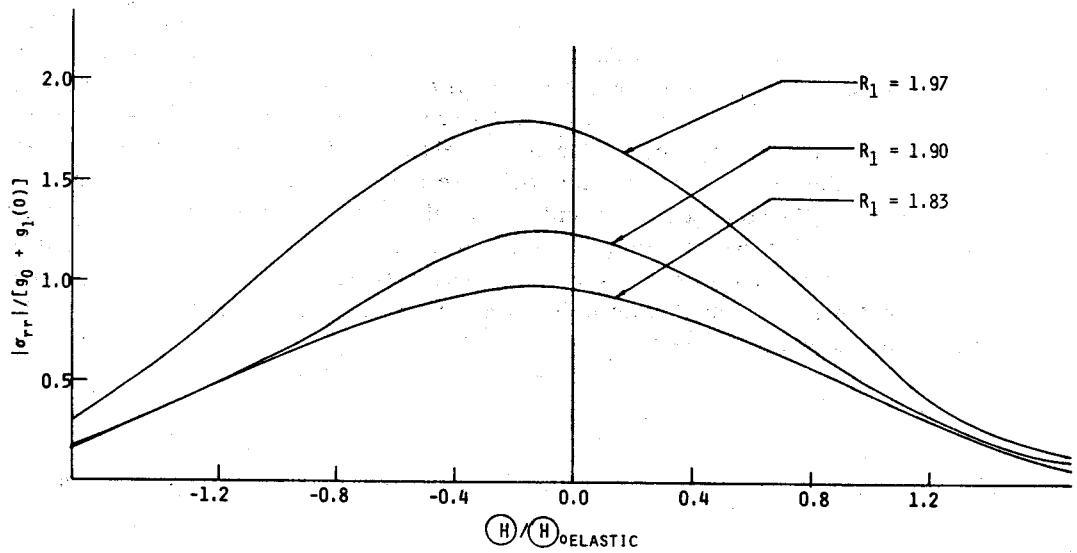


Figure 9. Radial stress distribution near the bond surface for different thicknesses of the rubberlike layer

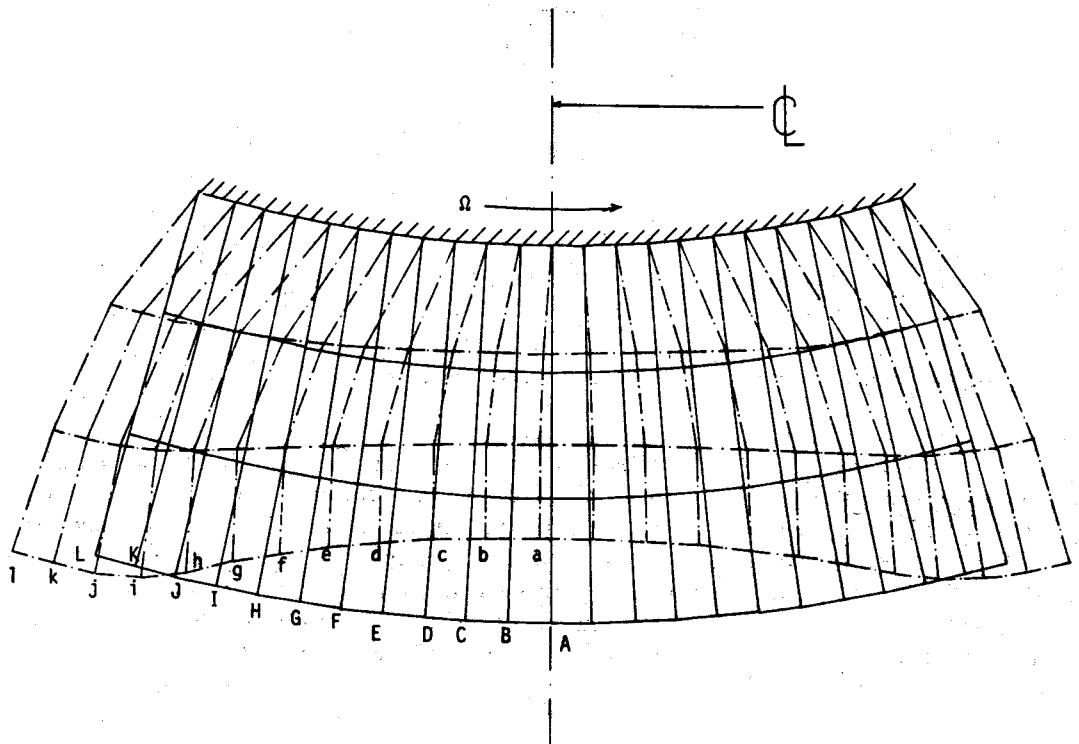


Figure 10. Details of the deformation of the rubber in the vicinity of the contact width

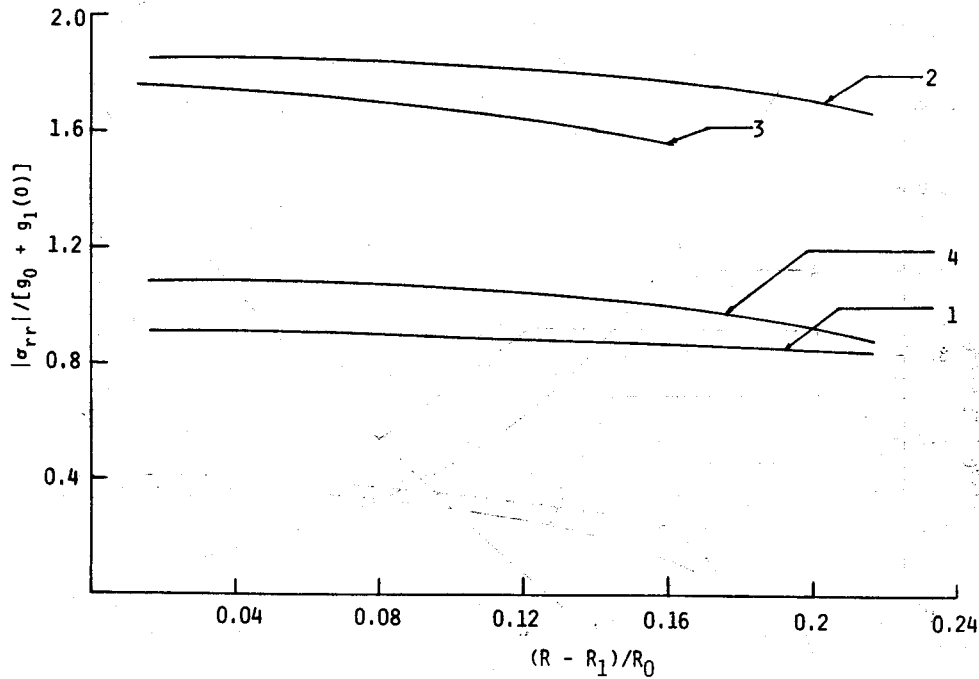


Figure 11. Variation of radial stress through the thickness of the rubberlike layer at $\Theta = -0.00552$ rad.

As is evident from the distribution of the radial stress plotted through the thickness of the roll cover (Figure 11), for a specific value of Θ , the maximum radial stress occurs at points of rubber on the bond surface. The value of the radial stress for a given set of values of the material and geometric parameters does not vary much through the thickness of the roll cover. In Figure 12 is plotted the load required to indent the rubber as a function of various parameters. The variation of the load with the change in the thickness of the rubber layer is rather predominant. Also significant is the dependence of the load P on the values of the material moduli. In these calculations the values of g_0 and $g_1(0)$ were such as to keep $(g_0 + g_1(0))$ fixed. Thus, as the value of $g_1(0)$ increases, the load required to cause the same indentation of the rubber decreases. At the risk of repetition, we remark that the values of the material and geometric parameters, except the one being varied, are taken from equation (25). The load increases with the speed of the roll covers, but the change is not really that dramatic. Also, the radius of the indenter has less noticeable effect on the load.

CONCLUSION

The steady-state problem of a nonlinear viscoelastic roll cover in smooth contact with a rigid roll has been solved numerically by the finite element method. Since the problem is too complex to be solved analytically, no comments can be made as to the closeness of the numerical solution obtained to the analytical solution of the problem. Also, due to the lack of the experimental data available in the public domain, no comparison has been made of the numerical solution with the experimental findings. However, the computed solution exhibits the behaviour expected intuitively. Also, results computed from this code for two problems studied previously by the finite element method compared favourably with those obtained by using different

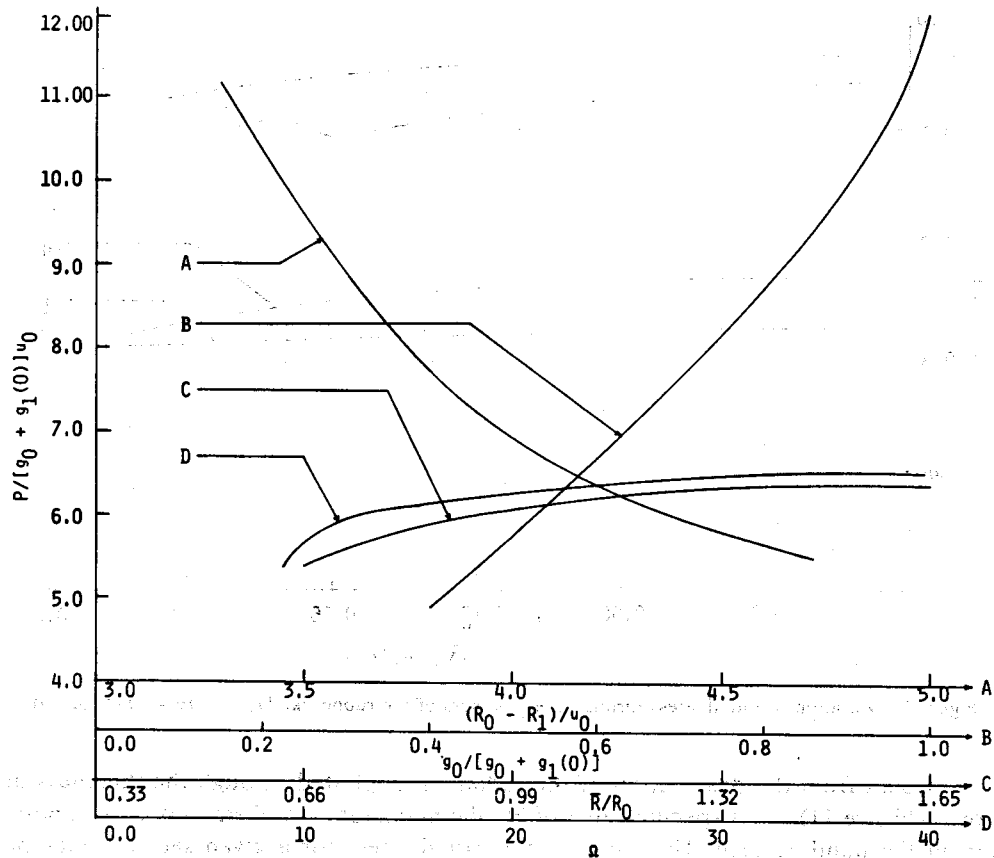


Figure 12. Effect of various parameters on the total load required

computer codes developed earlier. All this indicates that the presently developed computer code to solve the title problem yields reasonably good results.

It seems that the critical factors in determining the performance of these roll covers are the thickness of the rubber layer, the relative values of the viscoelastic modulus g_1 to the elastic modulus g_0 and, to a somewhat lesser extent, the speed of operation of the roll covers and the radius of the indenter.

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