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COLD SHEET ROLLING, THE THERMOVISCOELASTIC PROBLEM, A NUMERICAL SOLUTION

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SUMMARY

The coupled system of equations governing the thermomechanical deformations of a viscoelastic sheet while it is being cold rolled is solved numerically. The pertinent energy equation is solved by the finite difference method and the mechanical problem is solved by the finite element method using uniform first order rectangular elements. The developed computer program enables one to compute the complete deformation and temperature fields in the sheet. Results presented graphically include the temperature distribution, the stress distribution at the middle surface, the contact pressure distribution and the asymmetric surface deformation of the sheet.

INTRODUCTION

Problems involving viscoelastic rolling contact find technical applications in the fields of paper and plastic processing. These problems have been studied both analytically[1–4] and experimentally.[5] The complexity of viscoelastic boundary problems, especially of those in which the boundary surface where surface tractions are prescribed is not a material surface, forces one to resort to numerical methods such as the finite element method.[3,4] Earlier study[3] of the deformation of a viscoelastic sheet while it is being cold rolled assumes homothermal deformation field. Since mechanical properties of viscoelastic materials depend significantly upon temperature, the study of heat generated due to thermomechanical coupling and its effect upon the mechanical properties of the material is necessary.

Thermoviscoelastic boundary value problems have been studied by Cost[6], Oden and Armstrong[7], and Batra *et al.*[4] The reader is referred to the book by Oden[8] for other references on this subject. Of these, the problem studied in Reference 4 is a contact problem. These authors study the thermoviscoelastic problem of the indentation of paper mill roll covers by the finite element method and assume that the rolls are rotating at a uniform speed, the steady state has been reached, the effect of dynamic forces and of frictional forces at the contact surface on the deformation of the roll cover is negligible and the temperature of each material particle is constant in time. Thus the energy equation can be integrated over a cycle and this averaged equation is a non-homogeneous ordinary differential equation in temperature. A fourth order polynomial in the independent variable is fitted to the dissipation function (the non-homogeneous term in the energy equation) and the energy equation is integrated directly. The deformation of the roll cover is studied in two parts. First the layer is assumed to deform due to temperature changes and then the mechanical problem is solved by taking the thermally deformed state as the reference configuration and thermal stresses as the initial stresses. This procedure though valid for the roll cover problem because of its special geometry is not quite general. What one should do is that after having obtained a solution of the energy equation, one

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should account for thermal effects in the mechanical problem by introducing apparent body forces caused by thermal gradients. This is the approach followed here. The present problem differs from the ones treated in references 4 and 6 in that a material particle is not undergoing a cyclic deformation and thus one cannot consider the cycle averaged energy equation. Also, in the problem of sheet rolling, the temperature varies both along the sheet and across the thickness of the sheet so that the thermal problem is at least two-dimensional.

The present thermoviscoelastic study of the cold rolling of a viscoelastic sheet is an extension of the earlier viscoelastic study of Lynch[3] and the thermoviscoelastic study of a roll cover by Batra *et al.*[4]

FORMULATION OF THE PROBLEM

A schematic diagram of the system studied herein is shown in Figure 1. We assume that the rollers are made of a material considerably harder than the material of the sheet so that these can

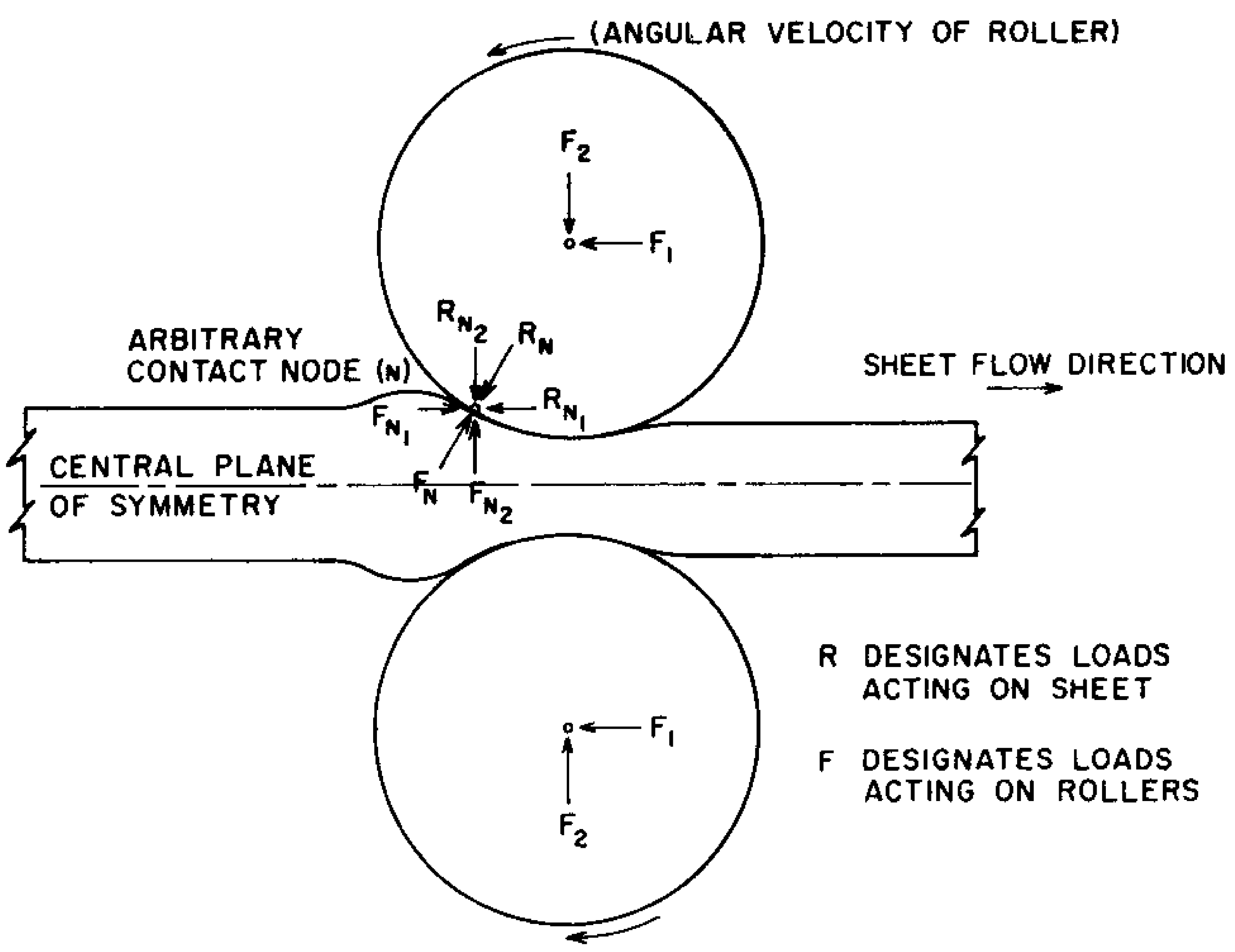

Figure 1. System to be studied

be regarded as being rigid. Assuming that there is no external body force and the supply of internal energy, thermomechanical deformations of the sheet are governed by the following system of equations[8]

$$\sigma_{ij,j} = \rho \ddot{x}_i, \qquad \rho \dot{e} = -q_{i,i} + \sigma_{ij}\dot{x}_{i,j}. \tag{1}$$

In (1) σ_{ij} is the Cauchy stress tensor, ρ is the present mass density, e is the internal energy per unit mass, $\mathbf{x}$ is the present position in rectangular Cartesian coordinates of a material particle $\mathbf{X}$ at time t, a superposed dot indicates material time differentiation, q_i is the heat flux measured per unit area in the present configuration, a comma followed by an index j indicates partial differentiation with respect to x_j, and the usual summation convention is used. (1) is to be supplemented by the constitutive relations for σ_{ij}, e and q_i and side conditions such as boundary conditions. Before we state these we outline below the assumptions made to simplify the problem.

We assume that the material of the sheet is homogeneous and isotropic, the sheet moves at a constant velocity v, steady state has been reached, contact between rigid rollers and the sheet is frictionless, and the deformations are small so that a constitutive law linear in displacement gradients and temperature gradients applies. Note that the present temperature of a material particle is not assumed to be close to some reference temperature so that constitutive relations need not be linear in the change of the temperature of a particle from the reference temperature. The reason for not making such an assumption is that the relaxation functions of the material of the sheet may be highly temperature dependent. Herein the shear modulus of the material of the sheet is taken to depend upon temperature in a non-linear manner whereas the bulk behaviour is assumed to be elastic and temperature independent. Such an assumption is not uncommon for viscoelastic materials.[3,4,9] Usually the value of the bulk modulus for such materials is about fifty times the value of the shear modulus at time $t=0$. This implies that the deformations of these materials are nearly isochoric so that the sum of the principal strains is close to zero. Accordingly we neglect terms involving squares of the sum of principal strains in the energy equation. Another simplifying assumption made is that v is presumed to be sufficiently small so that all inertia effects including the rate of change of temperature are negligible. Lastly we assume that the width of the sheet is very large and that plane strain state of deformation prevails.

Constitutive relations for σ_{ij}, $\mathbf{q}$ and e which are compatible with the Clausius–Duhem inequality are discussed in references 6, 9, 10. That for σ_{ij} is

$$\sigma_{ij}(\mathbf{X}, t) = \int_{-\infty}^{t} G_1(T, t-\tau)\frac{\partial \varepsilon_{ij}(\mathbf{X}, \tau)}{\partial \tau}\,\mathrm{d}\tau + \frac{\delta_{ij}}{3}\int_{-\infty}^{t}[G_2 - G_1(T, t-\tau)]\frac{\partial \varepsilon_{kk}(\mathbf{X}, \tau)}{\partial \tau}\,\mathrm{d}\tau - \alpha G_2(T-T_0)\delta_{ij}. \tag{2}$$

When the constitutive relations for $\mathbf{q}$ and e are substituted into $(1)_2$ and it is simplified further in view of the assumptions stated above, it becomes

$$0 = \kappa T_{,ii} + \Lambda \tag{3}$$

where

$$\Lambda(\mathbf{X}, t) = -\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial G_1}{\partial t}(t-\tau, t-\eta)\frac{\partial \varepsilon_{ij}(\mathbf{X}, \tau)}{\partial \tau}\frac{\partial \varepsilon_{ij}(\mathbf{X}, \eta)}{\partial \eta}\,\mathrm{d}\tau\,\mathrm{d}\eta, \tag{4}$$

$$\varepsilon_{ij}(\mathbf{X}, t) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{5}$$

$$u_i = (x_i - X_i). \tag{6}$$

Here κ is the constant thermal conductivity of the material, α is the constant coefficient of thermal expansion, G_1 and G_2 are, respectively, the shear and the bulk moduli, δ_{ij} is the kronecker delta, T is the present temperature of a material particle and T_0 is the temperature of a material particle when it was at minus infinity. Λ in (3) represents the energy dissipated because of the viscoelastic effects. For elastic materials G_1 is time independent and therefore $\Lambda = 0$. It should be noted that in (4) terms involving $\varepsilon_{ii}\varepsilon_{kk}$ have been neglected. Substitution of (4) into (3) and of (2) into (1) gives field equations for $\mathbf{x}$ and T. Note that these field equations are non-linear in temperature T and in $\mathbf{u}$. These field equations are to be solved under the following boundary conditions.

$$\sigma_{ij}n_j = 0, \qquad |x_1 + cl| \geq l, \qquad x_2 = 0, \tag{7}$$

$$\left.\begin{aligned} u_2(x_1, 0) &= d - \frac{x_1^2}{2R}, \\ e_i\sigma_{ij}n_j &= 0, \end{aligned}\right\} \quad |x_1 + cl| < l, \qquad x_2 = 0, \tag{8}$$

$$u_2 = 0, \qquad \sigma_{12} = 0, \qquad x_2 = D/2, \tag{9}$$

$$|\sigma_{ij}n_j| \to 0 \quad \text{as } |\mathbf{x}| \to \infty, \tag{10}$$

$$T \to T_0 \quad \text{as } |\mathbf{x}| \to \infty, \tag{11}$$

$$q_in_i = h(T - T_0), \qquad x_2 = 0, \tag{12}$$

$$q_in_i = 0, \qquad x_2 = D/2. \tag{13}$$

Here $\mathbf{n}$ is an outward directed unit normal to the bounding surface, $\mathbf{e}$ is a unit vector tangent to the bounding surface, D is the thickness of the sheet, d is the depth of indentation at $x_1 = 0$, R is the radius of the roller, cl is the distance between the contact centre and the centre line of the rollers as shown in Figure 2, $2l$ is the width of the contact surface and h is the surface heat

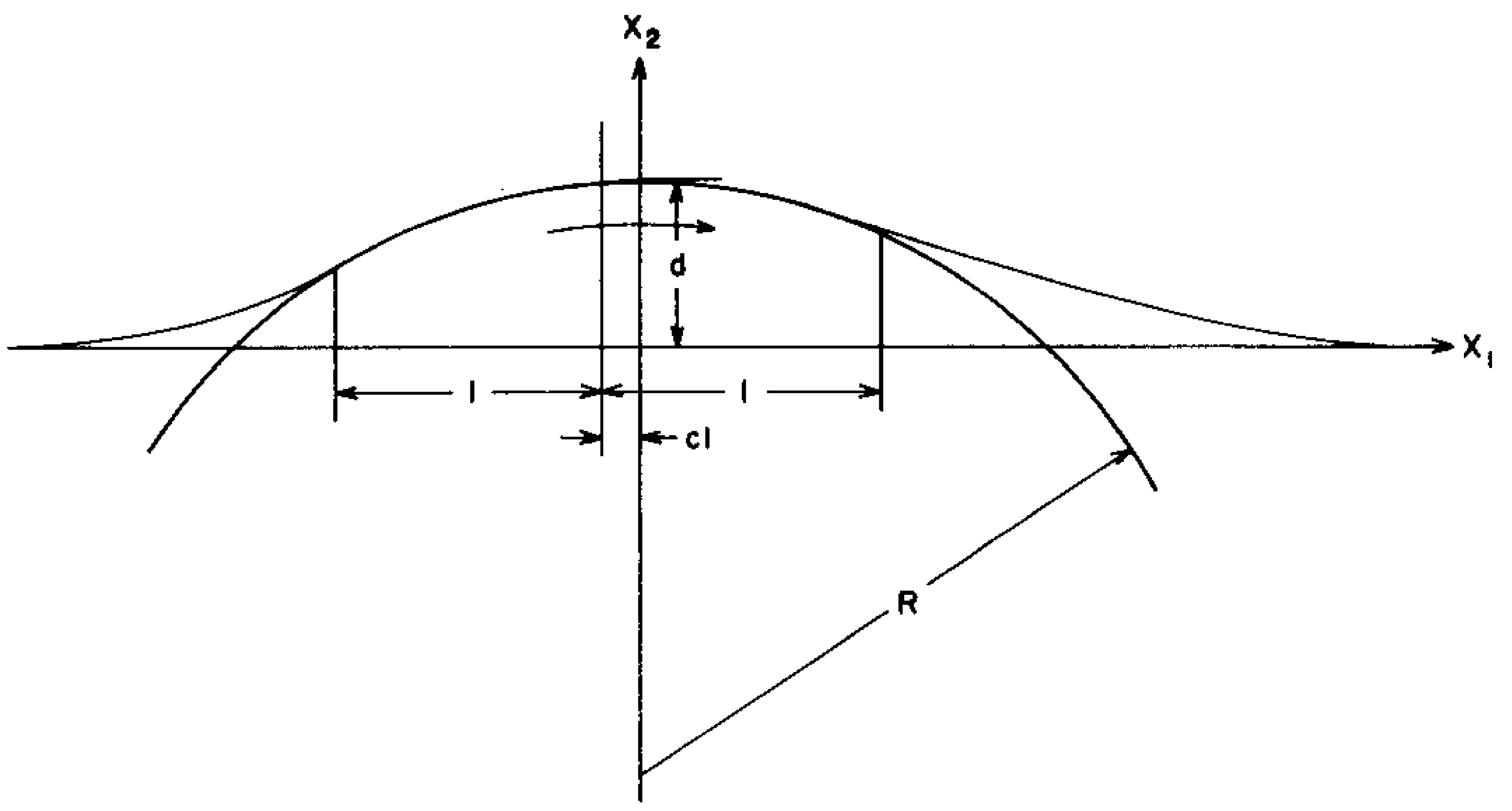

Figure 2. A configuration of the indented surface

transfer coefficient and its value depends upon the smoothness of the surface of the sheet. The fact that $\sigma_{ij}n_j = 0$ at $|x_1 + cl| = l$ implies that $n_i\sigma_{ij}n_j = 0$ and this ensures that the normal stress is continuous across the arc of contact and that a contact problem rather than a punch problem is being solved. Since a material particle is in contact with the surface of the roller for a fraction of a second, (12) appears to be a good approximation. If desired one can use different values of h for points on the contact surface. Of the three constants appearing in (7)–(13) only one can be assumed to be known and the other two are to be determined as a part of the solution.

The problem as formulated above is too difficult to solve analytically. Therefore we solve it numerically.

FORMULATION OF THE PROBLEM FOR A NUMERICAL SOLUTION

Because of the symmetry of the problem, we study the deformation of the lower half of the sheet. Motivated by the good agreement between the numerical results and the experimental observations reported in Reference 3 we limit our study to that portion of the sheet which at any instant occupies the region of space symmetric about the line joining the centres of the rollers and extending to about seven times the length of contact. This region of interest is divided into uniform rectangular elements as shown in Figure 3. We assume that the ends of this region are

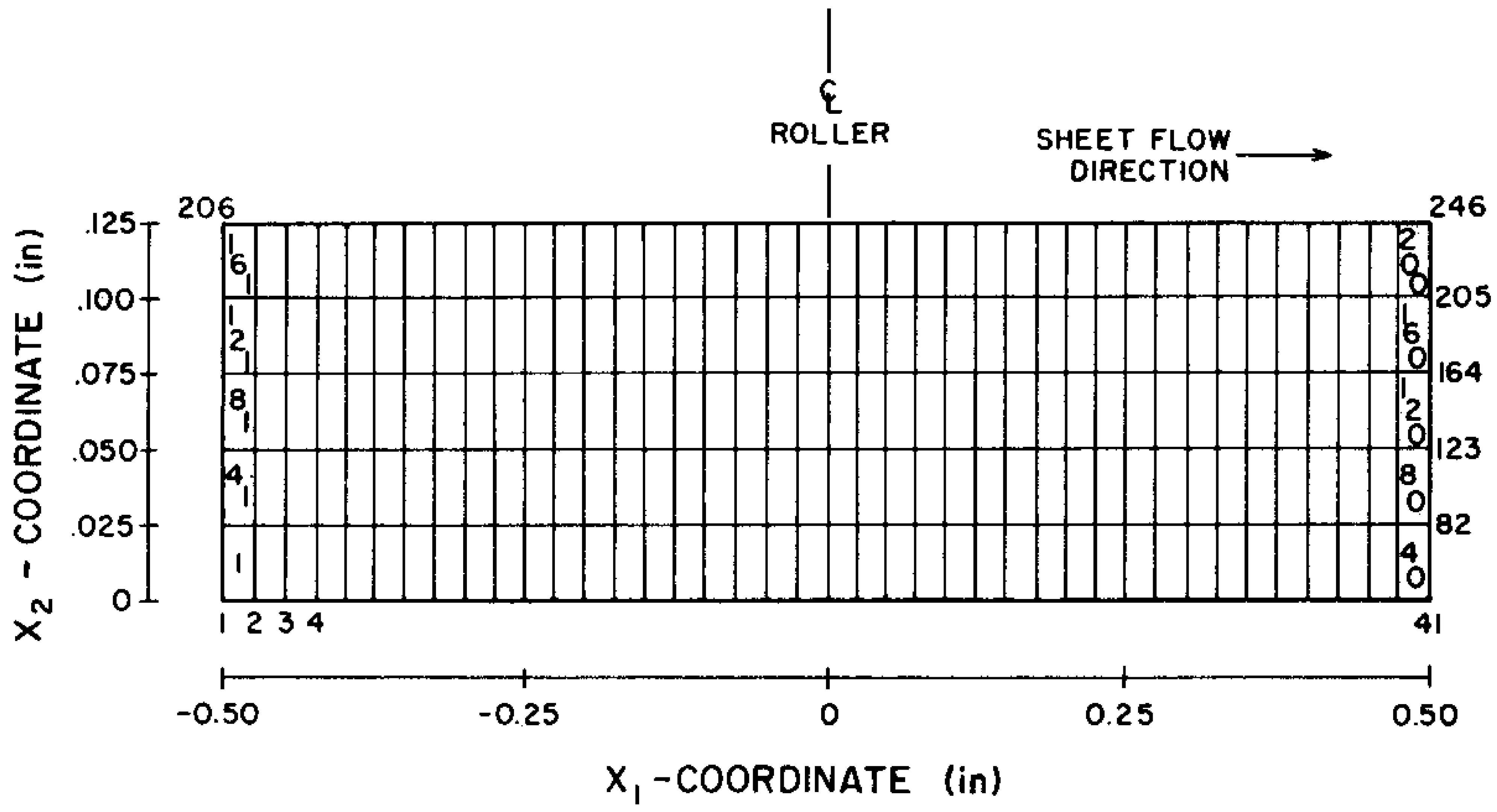

Figure 3. The grid

stress free and are at temperature T_0. In each element the displacement and temperature are assumed to be given by

$$\begin{aligned} u_1 &= f_1 + f_2 x_1 + f_3 x_2 + f_4 x_1 x_2, \\ u_2 &= g_1 + g_2 x_1 + g_3 x_2 + g_4 x_1 x_2, \\ T - T_0 &= \alpha_1 + \alpha_2 x_1 + \alpha_3 x_2 + \alpha_4 x_1 x_2, \end{aligned} \tag{14}$$

where f's, g's and α's are determined in terms of nodal displacements and nodal temperatures. Calculating strain components from the strain-displacement relation (5) we obtain

$$\{\boldsymbol{\varepsilon}\} = [\mathbf{A}]\{\boldsymbol{\delta}\} \tag{15}$$

where $[\mathbf{A}]$ is a 3×8 matrix whose elements are functions of nodal point co-ordinates, $\{\boldsymbol{\delta}\}$ is a vector of displacements of nodal points and $\{\boldsymbol{\varepsilon}\}$ is the vector of strain components ε_{11}, ε_{22}, ε_{12}. Taking $t = 0$ when a material particle enters the region of interest, we have

$$\varepsilon_{ij}(\mathbf{X}, t) = 0, \qquad t < 0.$$

We calculate stresses by approximating the integral in (2) by the sum of a finite series and writing it as

$$\{\boldsymbol{\sigma}\}_N = \sum_{R=1}^{N} [\mathbf{B}]_{NR}\{\boldsymbol{\varepsilon}\}_R + \{\boldsymbol{\beta}\}_N, \tag{16}$$

where

$$\{\boldsymbol{\sigma}\}_N = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_N ,$$

$$[\mathbf{B}]_{NR} = \begin{bmatrix} a_{NR} & b_{NR} & 0 \\ b_{NR} & a_{NR} & 0 \\ 0 & 0 & c_{NR} \end{bmatrix},$$

$$a_{NR} = \begin{cases} \frac{1}{3}(G_1(0) + G_1(\Delta t) + G_2), & R = N \\ \frac{1}{3}(G_1((N-R+1)\Delta t) - G_1((N-R-1)\,\Delta t), & R \neq N, \end{cases} \tag{17}$$

$$\{\boldsymbol{\beta}\}_N = -\alpha G_2 (T - T_0) \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix},$$

$$c_{NR} = (a_{NR} - b_{NR}),$$

$$\Delta t = 2A/v,$$

$2A$ = distance between the centroids of two consecutive elements, and b_{NR} is obtained from a_{NR} by replacing G_1 by $-\frac{1}{2}G_1$. Using the principle of virtual work

$$\{\mathbf{F}\}_N^T \{\boldsymbol{\delta}\}_N = \int_{V_N} \{\boldsymbol{\sigma}\}_N^T \{\boldsymbol{\varepsilon}\}_N \, \mathrm{d}V,$$

and substituting from (15) and (16), we obtain

$$\{\mathbf{F}\}_N = \sum_{R=1}^{N} [\mathbf{K}]_{NR} \{\boldsymbol{\delta}\}_R + \{\mathbf{H}\}_N, \tag{18}$$

where

$$[\mathbf{K}]_{NR} = \int_{V_N} [\mathbf{A}]_N^T [\mathbf{B}]_{NR} [\mathbf{A}]_R \, \mathrm{d}V,$$

$$\{\mathbf{H}\}_N = -\alpha G_2 \begin{Bmatrix} -\alpha_1 B + \frac{4}{3}\alpha_3 B^2 \\ -\alpha_1 A + \frac{4}{3}\alpha_2 A^2 \\ \alpha_1 B - \frac{4}{3}\alpha_3 B^2 \\ -\alpha_1 A - \frac{4}{3}\alpha_2 A^2 \\ \alpha_1 B + \frac{4}{3}\alpha_3 B^2 \\ \alpha_1 A + \frac{4}{3}\alpha_2 A^2 \\ -\alpha_1 B - \frac{4}{3}\alpha_2 B^2 \\ \alpha_1 A - \frac{4}{3}\alpha_2 A^2 \end{Bmatrix}_N ,$$

$$\alpha_1 = (T_i + T_j + T_k + T_m - 4T_0)/4,$$

$$\alpha_2 = (T_j + T_m - T_i - T_k)/4A,$$

$$\alpha_3 = (T_k + T_m - T_i - T_j)/4B,$$

and i, j, m, k signify the nodal points of a typical element in the counterclockwise direction starting from the lower left corner. In these equations, $\{\mathbf{F}\}_N$ is the column vector of forces acting at nodal points of the element N, V_N is its volume, and $2B$ denotes the length of the element along the x_2 axis. The α's in these equations are the same α's given in $(14)_3$. Equation (18) states that nodal forces on a typical element N in a control strip are affine functions of displacements of nodal points associated with all the preceding elements in the same strip. In carrying out the integration we use the value of G_1 corresponding to the temperature of the centroid of the element. The elements of $[\mathbf{K}]_{NR}$ are listed in Reference 4. Calculating forces for each element and assembling these for the entire grid we obtain

$$\{\mathbf{F}\} = [\mathbf{K}]\{\boldsymbol{\delta}\} + \{\mathbf{H}\}, \tag{19}$$

where $\{\mathbf{F}\}$ is the vector of nodal forces, $[\mathbf{K}]$ is the stiffness matrix for the entire grid, $\{\boldsymbol{\delta}\}$ is the vector of nodal displacements and $\{\mathbf{H}\}$ is the vector representing body forces induced by temperature gradients. (19) is the equation of equilibrium and is the finite element equivalent of $(1)_1$.

To cast (3) into a form suitable for numerical work we found it convenient and economical to use the finite difference method. Thus we write (3) as (e.g. see Reference 11, p. 134)

$$[\mathbf{A}]\{\mathbf{T}\} = \{\boldsymbol{\lambda}\} \tag{20}$$

where $\{\mathbf{T}\}$ is a column vector representing temperatures of nodal points, $[\mathbf{A}]$ is a banded matrix whose elements incorporate the flux boundary conditions (12) and (13) and $\{\boldsymbol{\lambda}\}$ is a vector representing the values of Λ at various nodal points except for nodal points on the bounding surface for which this value is to be suitably modified to account for the boundary conditions. The mesh used to write (3) in the form (20) is the same as that used in the finite element formulation of $(1)_1$.

In the results presented below, G_1 is taken as

$$\begin{aligned} G_1(T, t) &= g_0 + g_1\, e^{-t/\tau(T)}, \\ \tau(T) &= \tau(T_0)/f(T), \\ \log_{10} f(T) &= \frac{8{\cdot}86(T-T_0)/1{\cdot}8}{101{\cdot}6 + (T-T_0)/1{\cdot}8}. \end{aligned} \tag{21}$$

Here τ is the relaxation time which depends upon the temperature as indicated above. The assumptions $(21)_2$ and $(21)_3$ imply that the material is thermorheologically simple[10,12] and equations $(21)_{2,3}$ relating the relaxation time $\tau(T)$ at temperature T(°F) to the reference temperature T_0(°F) constitute the WLF equation.[12] In $(21)_3$ the factor 1·8 is the conversion factor from °F to °C and the reference temperature is taken equal to T_0 for convenience. Figure 4 depicts how G_1 depends upon temperture. Substituting from (21) into (4) and approximating the integral by a finite sum, we obtain for a nodal point situated at Mth position from the left end on a row in the interior of the grid,

$$\lambda_M = -\frac{g_1}{2\tau(T_M)J}\, e^{-[2(M-1)\Delta t/\tau(T_M)]} \sum_{R=2}^{M} e^{[2(R-1{\cdot}5)\Delta t/\tau(T_M)]} S_R, \tag{22}$$

where

$$S_R = [(\varepsilon_{ij})_R - (\varepsilon_{ij})_{R-1}][(\varepsilon_{ij})_R - (\varepsilon_{ij})_{R-1}].$$

In (22) J is the Joule's constant, T_M is the temperature at the nodal point and $(\varepsilon_{ij})_R$ is the value of strain component at the nodal point situated, respectively, at the Mth and the Rth position from the left end. We remark that (19) and (20) are non-linear in temperature and (20) is non-linear in ε_{ij} also.

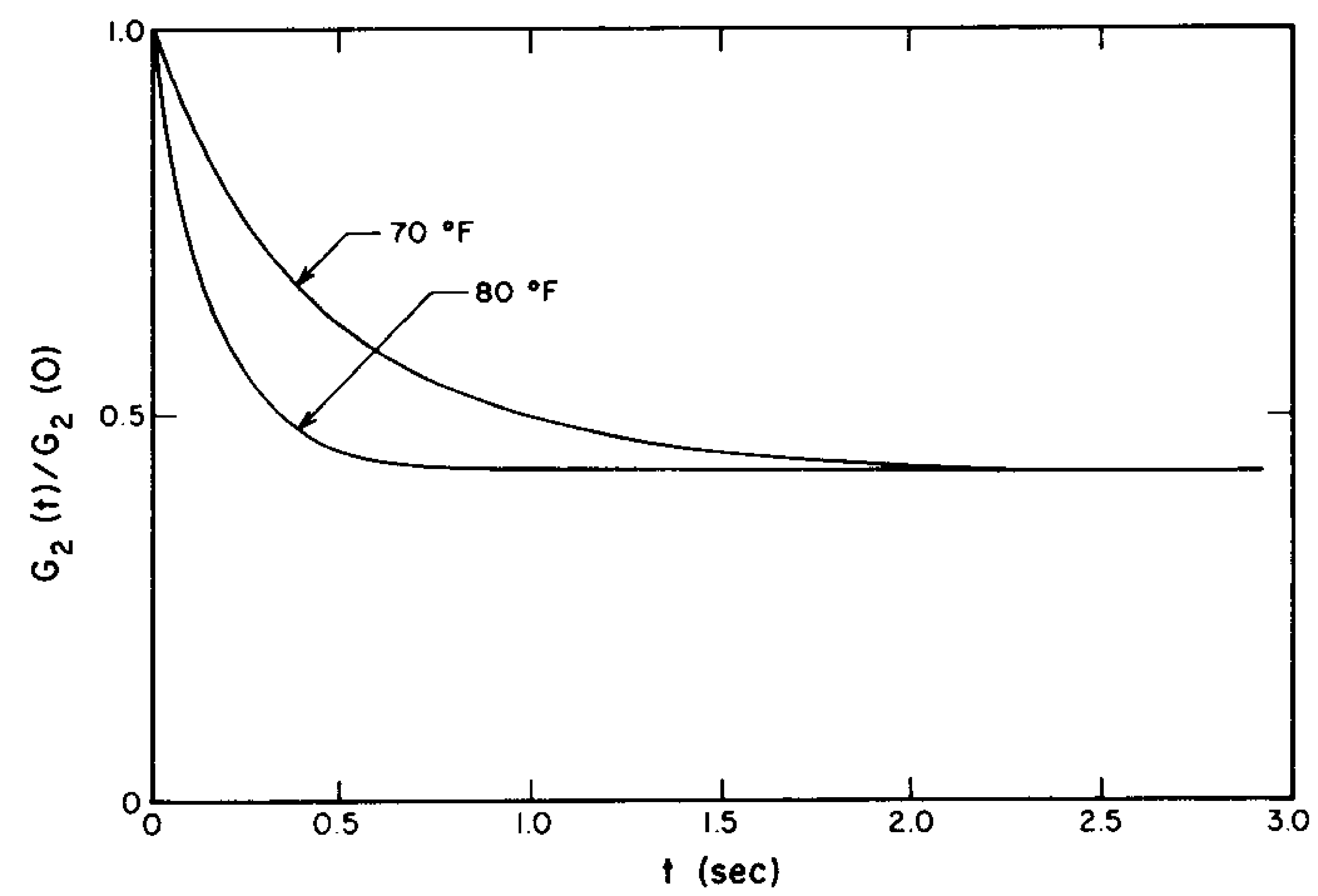

Figure 4. Variation of shear modulus with temperature

A solution of (19) and (20) satisfying various boundary conditions is found by the following iterative procedure. For the (zero)th iteration we take the temperature to be uniform throughout the region of interest and solve the homothermal viscoelastic contact problem. For this case $\{\mathbf{H}\}=\{\mathbf{0}\}$. To prevent rigid body motion in the x_1-direction, the x_1-component of the displacement of the central node point on the middle surface is set equal to zero. The mechanical contact problem is solved in a way essentially similar to that outlined in reference 4, except that now the value of d in (8) is prespecified. From this solution of the mechanical problem, values of displacements and strains are calculated and thence the right-hand side of (20) is computed. (20) is solved for T and the mechanical contact problem solved again by using this value of T. The whole process is repeated until, at each nodal point the difference in the values of temperature in two successive iterations is less than one per cent of the value of the temperature obtained in the immediately preceding iteration.

RESULTS FOR A SAMPLE PROBLEM

We took the following values of various geometric and material parameters to compute results for a hypothetical problem.

$$G_1(T,t)=5{,}200+6{,}800\,\mathrm{e}^{-t/\tau(T)}\ \text{psi}$$

$$G_2(T,t)=6\times10^5\ \text{psi},$$

$$\tau(70°\text{F})=0{\cdot}46\ \text{sec},\qquad T_0=70°\text{F},\qquad v=0{\cdot}4\ \text{in/sec},$$

$$\alpha=0{\cdot}00014/°\text{F},\qquad \kappa=0{\cdot}00856\ \text{BTU/hr in °F},$$

$$h=1{\cdot}5\ \text{BTU/hr ft}^2\ °\text{F},\qquad d=0{\cdot}00575\ \text{in},$$

$$D=0{\cdot}25\ \text{in},\qquad R=1{\cdot}25\ \text{in}$$

With a reasonably good estimate of the normal loads on the contact surface, the number of iterations required on the normal loads can be less than five for the solution of the mechanical problem to converge with the following criterion. The indented surface conforms to the circular profile of the roller if each nodal point on the contact surface lies within 0·01 d of the circular profile of the roller. With the above stated criterion for the convergence of the temperature the number of iterations required for the temperature field to converge was four. The Gaussian elimination followed by back substitution technique was used to invert the stiffness matrix and the total computation time including the compilation time etc. required for each iteration on an IBM 370 computer was 1·5 min. This is so because of the large band width of the stiffness matrix in the viscoelastic problem and the need to write the stiffness matrix on an auxiliary tape. For repeated use of the developed program it seems advisable to optimize it and there are places where it can be done.

Figure 5 shows the temperature distribution along the sheet at the free surface and at the middle surface. The temperature rise is not dramatic because of the low value of G_1 and v. At

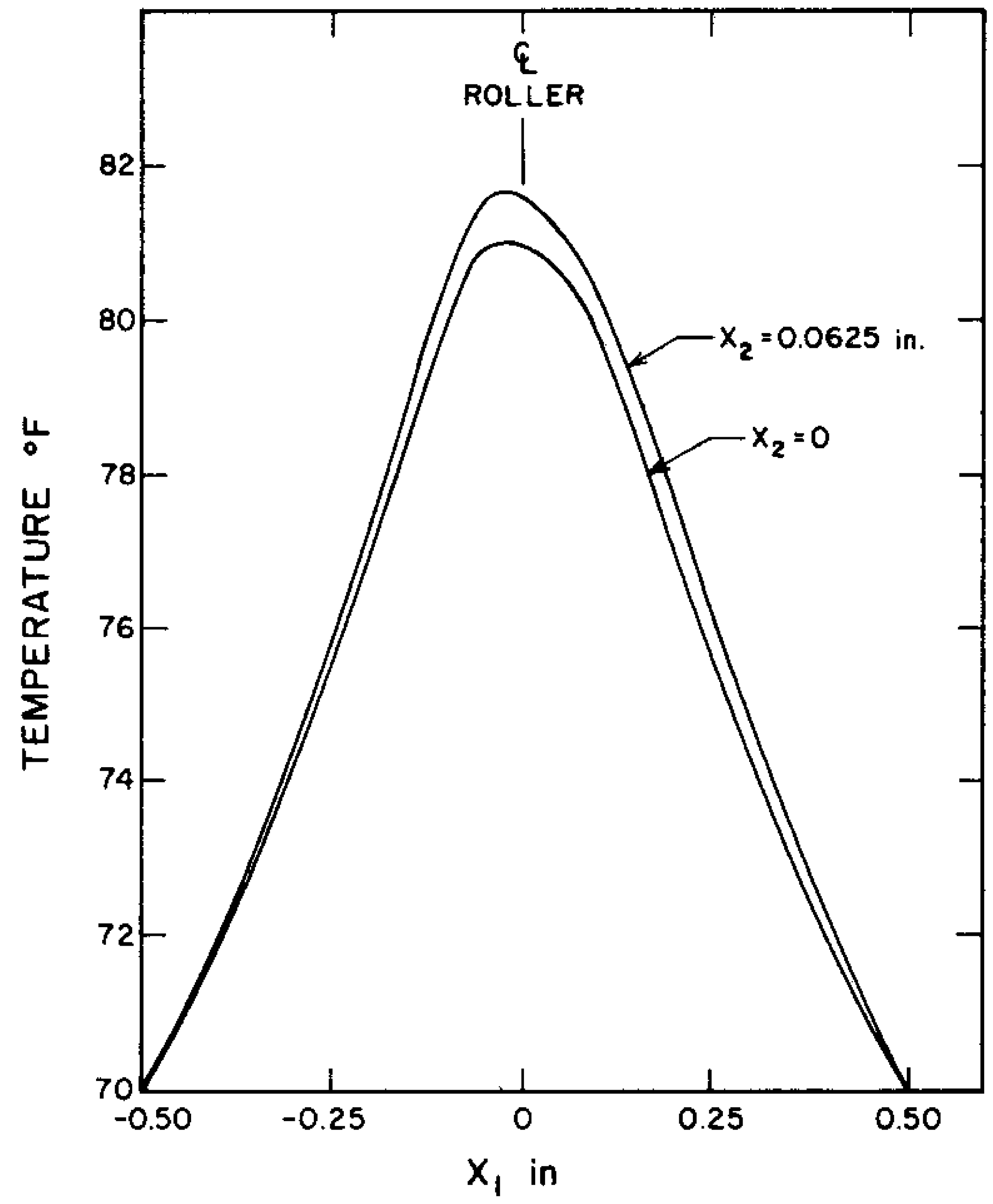

Figure 5. Temperature distribution along the sheet

any point along the sheet, the temperature was not found to vary much along the thickness of the sheet. For example along the centre line of the rollers the temperature at various nodal points starting from the one on the free surface came out to be 81, 81·3, 81·5, 81·6, 81·6, and 81·6 °F. As is clear from the figure, for a fixed value of x_2, the maximum value of temperature occurs at a point slightly to the left of the centre line of the rollers. This is due to the asymmetric deformation of the sheet and this asymmetry is apparent in Figures 6 and 7 which depict, respectively, the surface deformation of the sheet and the pressure distribution on the contact surface. The semi-contact width for the elastic case as found graphically from Figure 6 is 0·09 in. and that for the thermoviscoelastic case is 0·08 in. The value of the parameter c appearing in (7) also determined from Figure 6 is 0·153 for the thermoviscoelastic case. The total force required to cause the same indentation is smaller in the thermoviscoelastic case than that for the viscoelastic problem because the material is softened by the rise in temperature. The rise in temperature results in more asymmetry in the contact pressure distribution. The power required to drive the rollers when thermal effects are included is 1·31 of that for the homothermal

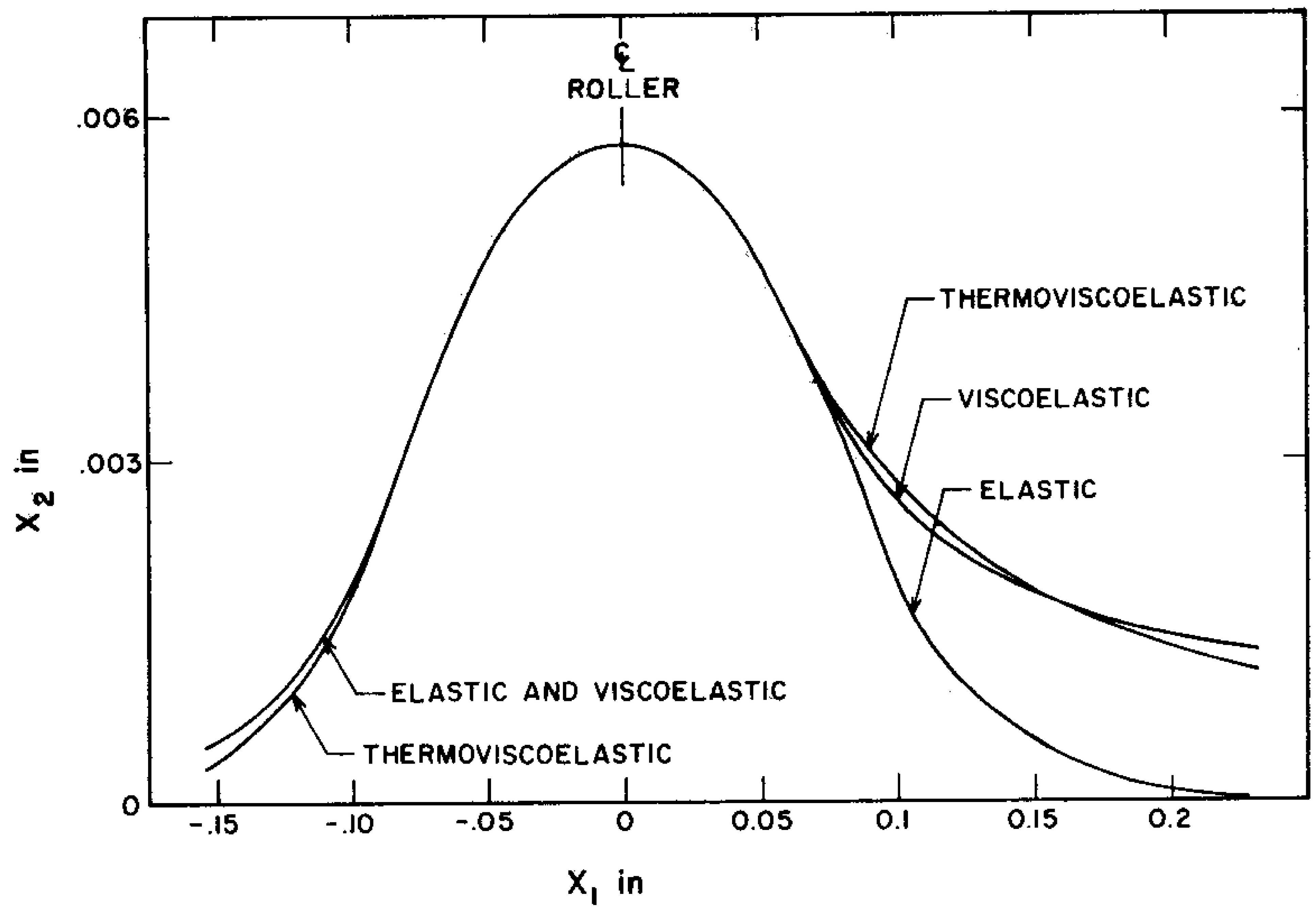

Figure 6. Asymmetric surface deformation of the indented surface

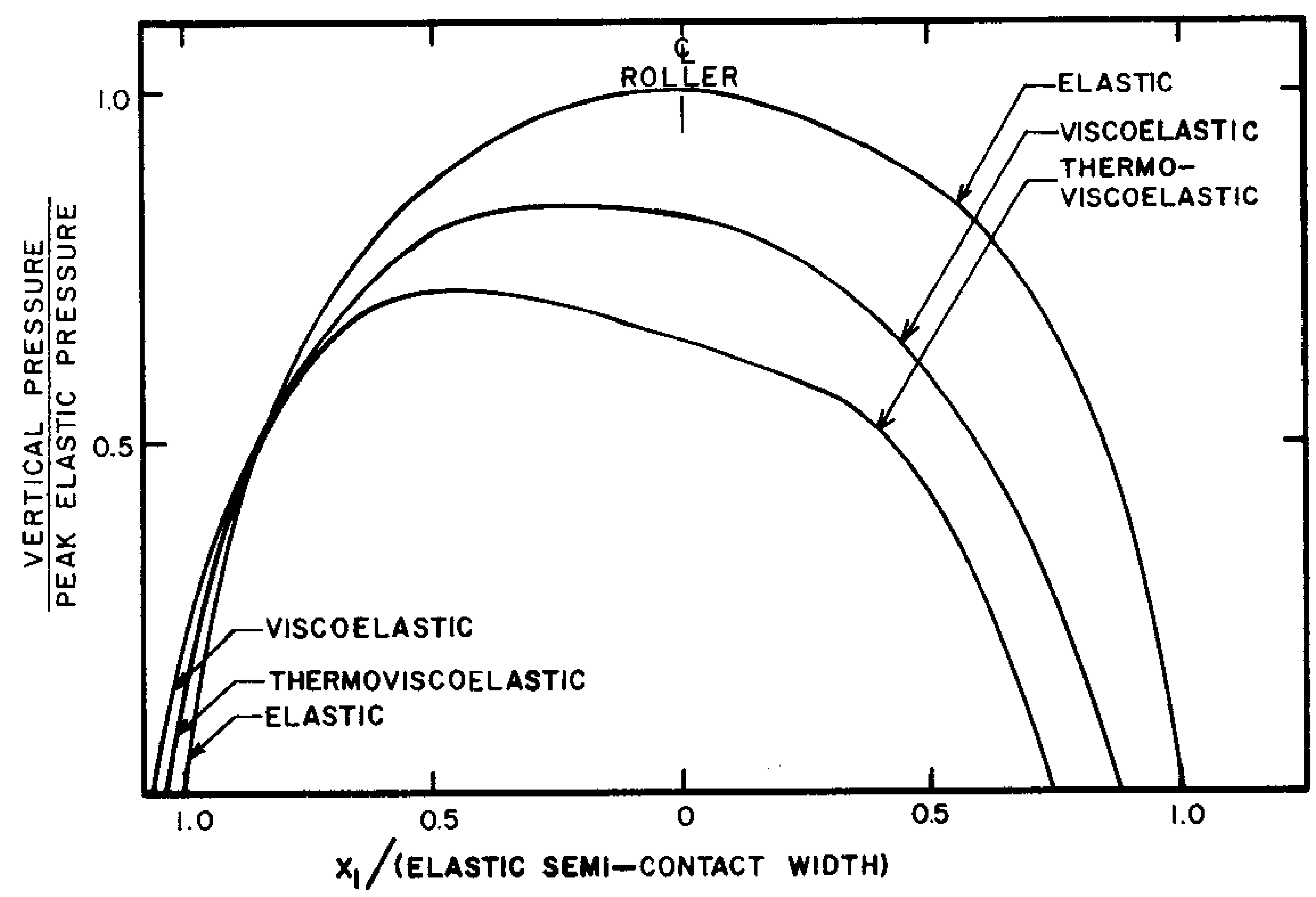

Figure 7. Comparison of thermoviscoelastic, viscoelastic and elastic contact pressure distributions

viscoelastic case. The deformed surface of the sheet in the thermoviscoelastic problem is below that for the viscoelastic problem because of the thermal expansion of the sheet in the former case. The thickness of the sheet when it leaves the region of the grid is 0·25 in., 0·2486 in. and 0·2486 in. for the elastic, viscoelastic and the thermoviscoelastic cases respectively. The reason for the thickness of the sheet to be the same in the viscoelastic and thermoviscoelastic problems is that the temperature at $x_1 = 0{\cdot}5$ in. is taken to be equal to the reference temperature which is the same as that at $x_1 = -0{\cdot}5$ in.

In Figure 8 is plotted the normal stress at the middle surface. Both elastic and homothermal viscoelastic analyses yield values for the maximum compressive stress greater than that obtained from the thermoviscoelastic analysis. The stress for the viscoelastic and the thermoviscoelastic cases is asymmetric about the centre line of the rollers; the compressive stress is maximum at a point 0·02 in. to the left of the roller in the thermoviscoelastic problem.

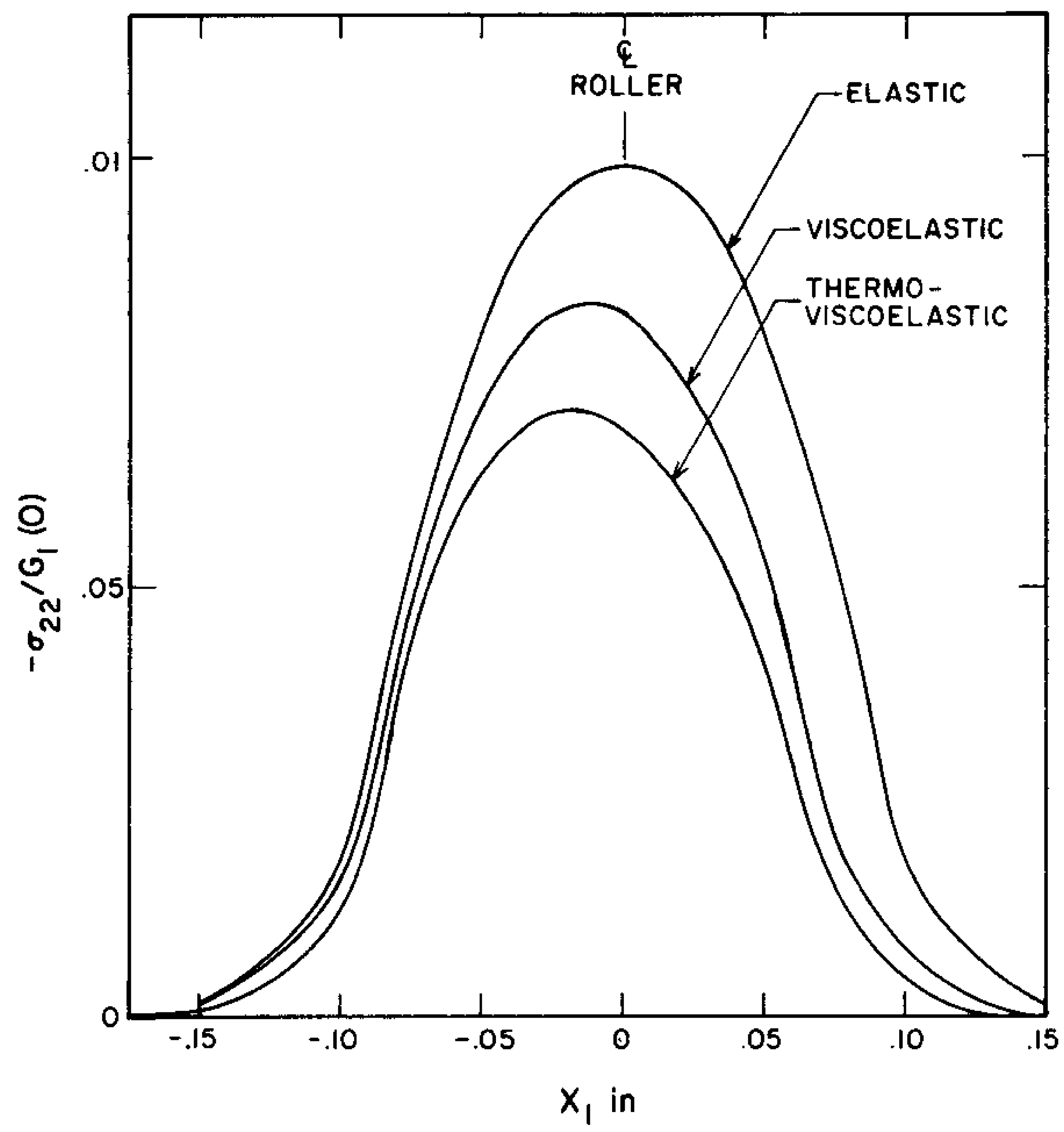

Figure 8. Normal stress distribution at the middle surface

REMARKS

The values of G_1 and G_2 taken for the sample problem are, respectively, twenty times the values of G_1 and G_2 for Polyvinyl butyral with 37·5 per cent by weight of dibutyl sebacate plasticizer reported in reference 3. The present results for the homothermal viscoelastic problem could not be compared with those obtained earlier by Lynch[3] because of the difficulty encountered in scaling off the data from the graphs presented in Reference 3 and more specifically in determining the value of the semicontact width at zero speed of the sheet. However the results do agree qualitatively.

Herein $G_1(T, 0)$ and G_2 are taken to be constants. If the dependence of $G_1(T, 0)$ upon temperature, and the viscoelastic bulk behaviour is to be accounted for only minor modifications of the analysis and the computational algorithm need to be made. Also the present work applies to the cold rolling of a slab of any thickness since nowhere do we require that the thickness be small. This work can be extended to include the frictional force at the constant surface, inertia effects, the dependence of the thermal conductivity and the coefficient of expansion upon temperature and the geometric and material non-linearities.

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