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Analysis of failure modes in an impact loaded thermoviscoplastic prenotched plate

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Abstract

Kalthoff observed experimentally that the failure mode in a prenotched maraging steel plate impact loaded on the notched side changes from brittle to ductile with an increase in the impact speed. Here we numerically investigate the effect on the failure mode transition speed of the shape of the notch-tip and the presence of a hole ahead of a circular notch-tip. The shape of the notch-tip is varied by changing the ratio, a/b, of the principal axes of an elliptic notch. For a circular notch-tip, we also investigate the effect, on the failure mode transition speed, of the presence of a circular void ahead of the notch-tip and situated either on or away from the axis of the notch. The Bodner-Partom thermoviscoplastic relation is used to model the strain hardening, strain-rate hardening and thermal softening response of the material of the plate. The transient plane strain thermomechanical deformations of the plate are analyzed by the finite element method. It is found that for a/b = 2.0 and 10.0, the brittle failure preceded the ductile failure for the six impact speeds studied herein, and for a/b = 0.4 and 1.0, the failure mode transitioned from the brittle to the ductile with an increase in the impact speed. The presence of a circular void ahead of the circular notch-tip shifts towards the axis of the notch the point on the the notch-tip surface where a shear band initiates. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Failure mode transition speed; Dynamic plane strain deformations, coupled thermomechanical problem; Finite element solution

1. Introduction

Kalthoff and Winkler (1987) and Kalthoff (2000) have experimentally studied the transient deformations of a prenotched maraging steel plate with the axes of the two

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notches parallel to the top and the bottom edges of the plate. The edge between the notches was struck by a cylindrical projectile of diameter equal to the distance between the notches. They found that, with an increase in the impact speed, the failure mode transitioned from a brittle failure in the form of a crack to a ductile failure in the form of an adiabatic shear band. Zhou et al. (1996a) conducted similar tests and found that only a shear band initiated from the notch-tip, propagated nearly along the axis of the notch and got arrested. They attributed the discrepancy in the observed response of the plate to the difference in the material properties of the two steels. The work described herein shows that the shape of the notch-tip strongly influences the failure mode. For an elliptic notch-tip with the principal axes of lengths 2a and 2b, the failure mode transitions from the brittle to the ductile failure with an increase in the impact speed when a/b = 0.4 and 1.0; the principal axis of length 2a is along the axis of the notch. However, the ductile failure always precedes the brittle failure for a/b=2 and 10. In the presence of a circular hole of radius r_0 located on the axis of the circular notch-tip of radius r_0 at distances of $3r_0$, $4r_0$, $5r_0$ and $6r_0$ from the center of the circular notch-tip, the brittle failure initiated first from a point on the surface of the circular hole. When the horizontal distance between the centers of the hole and the notch tip equalled $4r_0$ and the center of the hole was located either above or below the axis of the notch at distances of r_0 or $3r_0$, both failure modes ensued from points on the surface of the hole in the former case. In the latter configuration, the ductile failure initiated from a point on the notch surface and the brittle failure from a point on the surface of the circular hole.

We note that in the Charpy V-notch test, the ductile failure ensues at low loading rates or at high temperatures and the brittle failure at high loading rates or at low temperatures. Tvergaard and Needleman (1986, 1993) and Needleman and Tvergaard (1995) have numerically analyzed the Charpy and the Kalthoff tests by using Gurson's (1977) flow potential and a power law type relation to model the thermoviscoplastic response of a microporous steel. They showed that predictions from the same set of governing equations are in qualitative agreement with the findings of the two tests. The Kalthoff problem has also been numerically studied by Zhou et al. (1996b), Batra and Nechitailo (1997), Batra and Gummalla (2000), Batra and Ravisankar (2000) and Batra and Jaber (2001a,b). Whereas Batra and Ravisankar (2000) analyzed three-dimensional deformations of the prenotched plate and the cylindrical projectile, other investigations studied plane strain deformations of the plate. They employed different constitutive relations but obtained qualitatively similar results. Batra and Jaber (2001a) used the Johnson-Cook, the Bodner-Partom, the Litonski-Batra and a power law type relation to model the thermoviscoplastic response of the plate. They determined the values of material parameters by solving initial-boundary-value problems simulating the same simple shear test and used these to study thermomechanical deformations of the plate. They showed that results for the Kalthoff problem computed with the four constitutive relations agreed qualitatively but the failure mode transition speed depended upon the constitutive relation employed. Here, we use the Bondner and Partom (1975) relation to model the thermoviscoplastic response of the plate.

2. Formulation of the Problem

A schematic sketch of the problem studied is shown in Fig. 1. We neglect the effect of body forces and the sources, if any, of the internal energy. We use rectangular Cartesian coordinates to study transient thermomechanical deformations of the prenotched plate. These deformations are governed by the following balance laws of mass, linear momentum, moment of momentum and internal energy written in the referential description of motion.

$$(\rho J(1-f)) = 0, \tag{1}$$

$$\rho_0(1-f_0)\dot{v}_i = T_{i\alpha,\alpha},\tag{2}$$

$$T_{i\alpha}x_{i,\beta} = T_{i\beta}x_{i,\alpha},\tag{3}$$

$$\rho_0(1-f_0)\dot{e} = -Q_{\alpha,\alpha} + T_{i\alpha}\dot{x}_{i,\alpha}.$$
(4)

Here, ρ is the present mass density of a material point whose mass density in the reference configuration is ρ_0 , *f* the present porosity, a superimposed dot indicates the material time derivative, **v** the velocity, **T** the first Piola–Kirchhoff stress tensor, **x** the present position of the material particle that occupied the place **X** in the reference configuration, $x_{i,\beta} \equiv \partial x_i/\partial X_\beta$, $J = \det[x_{i,\alpha}]$, a repeated index implies summation over the range of the index, *e* is the specific internal energy, and **Q** the heat flux per unit reference area.

The plate is assumed to be made of a homogeneous and isotropic material, and its thermomechanical response modeled by the following constitutive relations.

$$\dot{\sigma}_{ij} - \Omega_{ik}\sigma_{kj} + \sigma_{ik}\Omega_{kj} = \frac{E(1-f)}{(1+\nu)}D^e_{ij} + \frac{\nu E(1-f)}{(1+\nu)(1-2\nu)}D^e_{kk}\delta_{ij},$$
(5)

$$D_{ij} = D_{ij}^{e} + D_{ij}^{p} + \alpha \dot{\theta} \delta_{ij}, \quad D_{ij} = (v_{i,j} + v_{j,i})/2, \quad \Omega_{ij} = (v_{i,j} - v_{j,i})/2, \quad (6)$$

$$D_{ij}^{p} = \frac{(1-f)\sigma_{e}\dot{\varepsilon}^{p}}{\sigma_{kl}N_{kl}}N_{ij}, \quad N_{ij} = \frac{\partial\Phi}{\partial\sigma_{ij}},\tag{7}$$

$$\Phi \equiv \frac{3}{2} \frac{(s_{ij} s_{ij})}{\sigma_e^2} + 2f^* q_1 \cosh\left(\frac{q_2 \sigma_{kk}}{2\sigma_e}\right) - 1 - q_1^2 f^{*2} = 0,$$
(8)

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij},\tag{9}$$



Fig. 1. (a) A schematic sketch of the problems studied; (b) an elliptic notch-tip; (c) a view of a circular notch-tip and a circular hole located directly ahead of the notch; (d) a view of a circular notch-tip and a circular hole located away from the axis of the notch.

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$$\dot{\varepsilon}^p = D_0 \exp\left[-\frac{1}{2}\left(\frac{K^2}{3\sigma_e^2}\right)^n\right], \quad \sigma_e = \left(\frac{3}{2}\sigma_{ij}\sigma_{ij}\right)^{1/2},\tag{10}$$

$$K = K_1 - (K_1 - K_0) \exp(-mW_p), \quad n = \frac{A}{\theta} + B, W_p = \int_0^t \sigma_{ij} D_{ij}^p dt,$$
(11)

$$f^* = \begin{cases} f \text{ if } \leq f_c \\ f_c + \left(\frac{f_u - f_c}{f_f - f_c}\right)(f - f_c), \text{ if } f > f_c \end{cases}$$
(12)

$$\dot{f} = (1 - f)D_{ii}^{p} + \frac{f_{2}\dot{\varepsilon}^{p}}{s_{2}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\varepsilon^{p} - e_{N}}{s_{2}}\right)^{2}\right),$$
(13)

$$Q_{\alpha} = -\kappa \left(1 - \frac{3}{2}f\right)\theta_{,\alpha},\tag{14}$$

$$\dot{e} = c\dot{\theta} + \sigma_{ij}D^{e}_{ij} + \alpha\dot{\theta}\sigma_{kk}.$$
(15)

$$T_{i\alpha} = J\sigma_{ij}X_{\alpha,j}.\tag{16}$$

Here, σ is the Cauchy stress tensor related to the first Piola–Kirchhoff stress tensor by (16), Ω the spin tensor, E Young's modulus, v Poisson's ratio, α the coefficient of thermal expansion, D the strain-rate tensor, $\dot{\varepsilon}^p$ the effective plastic strainrate, k the thermal conductivity, and c the specific heat. Other parameters characterizing the material are $q_1, q_2, f_2, s_2, e_N, f_c, f_u, D_0, K_1, K_0, m, A$, and B. The lefthand side of Eq. (5) is the Jaumann derivative of the Cauchy stress, Φ is the Gurson's (1977) flow potential for a microporous isotropic material as modified by Tvergaard and Needleman (1984), Eq. (10) is the Bodner and Partom (1975) relation used here to characterize the thermomechanical deformations of the material, and Eq. (14) is the Fourier law of heat conduction in which the thermal conductivity is taken to decrease affinely with an increase in the porosity (e.g. see Budiansky, 1970). The affine degradation of the Young's and the bulk moduli with an increase in the porosity has been proposed, amongst others, by Passman and Batra (1984) and Kobayashi and Dodd (1989). Eq. (13) for the evolution of the porosity is taken from Chu and Needleman's (1980) work; we have not considered the stress controlled nucleation of voids.

For the initial conditions, we take

$$\sigma(\mathbf{x}, 0) = \mathbf{0}, f(\mathbf{x}, 0) = 0, \rho(\mathbf{x}, 0) = \rho_0, \mathbf{v}(\mathbf{x}, 0) = \mathbf{0}, \theta(\mathbf{x}, 0) = \theta_0$$

That is, the plate is initially at rest, stress free and at a uniform temperature θ_0 . The following boundary conditions are imposed on the surface of the plate impacted by the cylindrical rod.

$$v_1/v_0 = \begin{cases} 0.3t, & 0 \le t \le 2\,\mu s, \\ 0.525 + 0.0375t, & 2 \le t \le 10\,\mu s, \\ 0.9, & 10\,\mu s \le t \le t_s \\ 0, & t > t_s, \end{cases}$$
(17)
$$T_{21} = 0, \ Q_1 = 0.$$

Here, v_0 is the speed of the projectile, v_1 the velocity of plate particles in the x₁-direction, and t_s equals the time when the projectile separates from the plate. This expression for v_1 is obtained by fitting straight lines to the data of Batra and Ravisankar (2000) who studied three-dimensional deformations of the plate and the projectile. Eqs. (17)₂ and (17)₃ imply that the contact surface is smooth and thermally insulated. All remaining bounding surfaces of the plate, including those of the notch and the circular hole if there is one, are assumed to be traction free and thermally insulated.

3. Computation and discussion of results

We assume that the material of the plate is HY 100 steel, and assign following values to various material and geometric parameters.

$$\sigma_{0} = 702 \text{ MPa}, \quad \rho_{0} = 7860 \text{ kg/m}^{3}, \quad c = 473 \text{ J/kg} \,^{\circ}\text{C}, \\ k = 49.73 \text{ W/m}^{2} \,^{\circ}\text{C}, \quad \alpha = 11.5 \times 10^{-6} /^{\circ}\text{C}, \\ E = 208 \text{ GPa}, \quad v = 0.3, \quad f_{2} = 0.04, \quad e_{N} = 0.3, \\ s_{2} = 0.1, \quad q_{1} = 1.5, \quad q_{2} = 1.0, \quad f_{c} = 0.15, \quad f_{f} = 0.25, \quad f_{u} = 0.667, \quad f_{0} = 0 \\ \theta_{0} = 25 \,^{\circ}\text{C}, \quad A = 1200 \text{ K}, \quad B = 0, \quad K_{1} = 2950 \text{ MPa}, \quad K_{0} = 2937 \text{ MPa} \\ m = 3120/\text{MPa}, \quad D_{0} = 1.732 \times 10^{8}/s, \quad v_{0} = 30 \text{ m/s} \end{cases}$$
(18)

Values of A, B, K_1 , K_0 , m and D_0 were obtained by finding the numerical solution of the initial-boundary-value problem closely simulating the torsional test conditions of Marchand and Duffy (1988) and comparing the computed shear stressshear strain curve with the experimental one (e.g. see Batra and Kim, 1990a). In (18), σ_0 equals the yield stress of the material in a quasistatic simple tension or compression test. Its value does not affect the solution of the initial-boundary-value problem being studied herein but is used in the definition of the brittle failure criterion, and to normalize the computed stresses. Note that according to the Bodner– Partom relation (10), the plastic strain rate is positive even when the effective stress is less than the pertinent yield stress of the material. Values of all of the thermophysical material parameters are likely to depend upon the temperature; such dependencies have been ignored here for the sake of simplicity.

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The three-dimensional analysis of the problem by Batra and Ravisankar (2000) revealed that deformations of the central 75% of the thickness of the plate closely correspond to the plane strain state of deformations. Here we presume that a plane strain state of deformation prevails in the plate. Because of the symmetry of the problem about the horizontal centroidal axis, we study deformations of only the upper half of the plate.

The aforestated problem is solved numerically by using an in-house developed finite element code. It employs constant strain triangular (CST) elements, lumped mass matrix derived by the row sum technique, and the subroutine LSODE (Livermore Solver for Ordinary Differential Equations) for integrating the stiff set of coupled nonlinear ordinary differential equations (ODEs). The Galerkin approximation of Eq. (2) and that obtained by substituting from (14)–(16) and (5)–(7) into (4) yields the ODEs for the determination of the nodal values of v_1 , v_2 and θ . Eqs. (1), (5), (10) and (13) are integrated at the centroids of the elements. Thus the number of ODEs to be integrated equals 3 (nodes)+7 (elements). A reasonably fine mesh was employed in the region around the notch-tip and the hole, and a coarse mesh elsewhere (see Fig. 2). Results presented herein have been computed with a fixed mesh.

Ritchie et al. (1973) have proposed that a brittle failure ensues when $\sigma_p/\sigma_0 = 3.0$ over a certain length that is characteristic of the microstructure of the material and typically equals the grain diameter; $\sigma_{\rm p}$ equals the maximum principal tensile stress. The size of the smallest CST element used here equals 23 μ m, the stresses are constant over an element, and the grain diameter is usually smaller than 23 μ m, thus the condition of the principal tensile stress exceeding a critical value is easily met. Based on the test data of Hendrickson et al. (1958) for tensile experiments on prenotched steel (yield strength = 705 MPa) specimens at nominal stress rates of approximately 1 to 10^4 MPa/s, we assume that a brittle failure initiates at the centroid of an element when $\sigma_p/\sigma_0 = 2.34$ there. Based on numerical experiments on the initiation and the development of adiabatic shear bands (ASBs), Batra and Kim (1992) proposed that an ASB initiates at a point when the shear stress there has dropped to 90% of its maximum value and the material point is deforming plastically; these and other numerical tests are summarized in Batra (1998). We take the initiation of the ductile failure as synonymous with the ensuing of an ASB and replace the shear stress by the effective stress. We emphasize that the computed failure mode transition speed will depend upon the criteria adopted for the initiation of the two failure modes. Our simulations neither included the opening of a crack nor the erosion of a failed element.

In their numerical study of the initiation and propagation of an ASB in a thermoviscoplastic body deformed in simple shear, Batra and Kim (1990b) found that an unloading elastic wave emanated out of the shear banded region and propagated outwards with a speed of $\sqrt{\mu/\rho}$. Here μ is the shear modulus of the material of the body. However, no such unloading wave emanated when the material response was modeled by a strain-rate-gradient dependent constitutive relation.

During the late stages of the postlocalization process, Batra and Chen (2001) have shown through numerical computations that in a typical steel nearly 85% of the



Fig. 2. A finite element discretization of the small region around the notch-tip and the circular hole for the three types of problems studied.

heat generated due to plastic working is conducted out of the edges of a shear band. Both these studies employed a mesh considerably finer than the one used herein.

3.1. Effect of the Shape of the Notch-tip

In linear elastic fracture mechanics, a sharp notch with essentially zero radius is considered. However, in experimental fracture mechanics, a sharp notch of zero radius is nearly impossible to achieve. We note that Kalthoff (1987, 2000) examined deformations of prenotched plates with circular notch-tips of radii exceeding 0.15 mm, and the radius of the circular notch-tip equalled 0.15 mm in the prenotched plates studied by Zhou et al. (1996a,b). Here we consider elliptic notch-tips with principal axes of lengths 2a and 2b, respectively; b = 0.15 mm for all problems studied, and the principal axis of length 2a is along the axis of the notch. A sharp notch is simulated by setting a/b = 10.0 and a blunt one by taking a/b = 0.4. Thermomechanical deformations of a prenotched plate with a/b = 0.2 could not be studied for t > 10 µs because of the severe concentrations of the deformations near the top and the bottom corners of the notch-tip. Unless otherwise stated, results presented and discussed below are for an impact speed of 30 m/s. For notch-tips with a/b = 0.4, 1.0, 2.0 and 10.0 we have plotted in Fig. 3a-c the fringes of the effective plastic strain, the normalized shear stress σ_{12} and the non-dimensional maximum principal stress in a small region surrounding the notch-tip. In each case, results are plotted at the instant of the initiation of the ductile failure which is the same as the time when the ASB initiates. The times of initiation of the ASB equal 12.30, 17, 16 and 13.20 μ s a/b = 0.4, 1.0, 2.0 and 10.0, respectively. Note that the time is reckoned from the instant of impact. It is clear that a deviation of the shape of the notch-tip away from the circular one results in an earlier initiation of the ASB in the impact loaded prenotched plate. For each notch-tip shape, the ASB initiates from a point on the surface of the notch-tip that is closer to the impacted edge. In the deformed configuration the angle measured clockwise between the axis of the notch-tip and the line joining its center to the point where the ASB initiates equals 85, 45, 12 and 3° , respectively, for a/b = 0.4, 1.0, 2.0 and 10.0. Thus, with an increase in the value of a/b, the point of initiation of the ductile failure shifts toward the center of the notchtip. At the instant and the location of the initiation of the ASB, the temperature rise, the effective plastic strain and the effective plastic strain-rate equal (158 °C, 0.365, 1.8×10⁵/s), (157 °C, 0.378, 0.42×10⁵/s), (122 °C, 0.293, 0.4×10⁵/s) and (82 °C, 0.175, 0.5×10^{5} /s), respectively, for a/b = 0.4, 1.0, 2.0 and 10.0. A typical value of the porosity at these four points equals 0.002 at the instant of the initiation of the ASB. The maximum strain induced nucleation rate of voids when the ASB initiates equals [cf. Eq. (13)] 8000/s and is smaller at other times. Since the computations were ceased once an element got severely deformed which occurred soon after the initiation of the ASB, there was not enough time in our computations for the porosity to evolve noticeably. For each value of a/b, the region with relatively large values of the effective plastic strain has two lobes; one essentially parallel to the axis of the notch and the other inclined at approximately 130° clockwise from the axis of the notch. However, for each one of the four values of a/b considered, the narrow region of the



Fig. 3. For the four elliptic notch-tips and the impact speed of 30 m/s, fringe plots of (a) the effective plastic strain, (b) the normalized shear stress σ_{12} , and (c) the normalized maximum principal stress in a small region surrounding the surface of the notch-tip. The times at which results are plotted equal respectively 12.3, 17, 16 and 13.3 µs for a/b = 0.4, 1.0, 2.0 and 10.0.



Fig. 3. (Continued)



Fig. 3. (Continued)



Fig. 3. (Continued)



24.8 24.6 49.6 49.8 50.0 50.2 50.4 50.6 **x₁-Coordinate(mm)**

Fig. 3. (Continued)



Fig. 3. (Continued)

maximum value of $|\sigma_{12}|$ is essentially parallel to the axis of the notch. It suggests that a shear band propagating along the axis of the notch is more intense than the other one. Fringe plots of the normalized maximum principal stress indicate that for each value of a/b the maximum principal tensile stress occurs at a point situated a little away from the upper surface of the notch-tip. For each one of the four notch-tips, the maximum principal stress is compressive in the region enclosing the point of initiation of the ductile failure. We note that the results are qualitatively similar for the four notch-tips.

Fig. 4 shows the deformed shapes of the four notch-tips and the locations in the deformed configurations of the points from where the brittle and the ductile failures initiate. Because of the Poisson effect, the lower surface of the notch moves upwards. Note that the upper surface is displaced very little. Also, points on the lower surface of the notch move to the right, i.e. in the direction of impact, thereby inducing tensile circumferential strains in the upper part of the notch surface. The two points where the brittle and the ductile failures initiate move towards the axis of the notch-tip with an increase in the value of a/b.

Figs. 5a and b exhibit the angular variation, at the time of initiation of the brittle failure, of the effective plastic strain and the normalized maximum principal stress at the centroids of elements abutting the surface of the notch-tip. Recall that the maximum principal tensile stress occurs at a point slightly away from the surface of the notch-tip. For an impact speed of 30 m/s, the ductile failure preceds the brittle failure for the four shapes of the notch tip. The angular positions in Fig. 5a and b are in the reference configuration. Out of the four values of a/b studied, values of the maximum effective plastic strain for a/b=0.4 and 1.0 are considerably higher than those for a/b = 2.0 and 10.0; however, the angular width of the intensely deformed region wherein the effective plastic strain equals at least 90% of the maximum value is smaller for a/b = 0.4 than that for the other three cases. Because of the elliptic shape of the notch-tip, the angular width is not proportional to the arc length for the four shapes of the notch-tip. For a/b = 1.0 and 2.0, the point-to-point variations in the values of the maximum principal stress are very small except near the notch-tip where it changes from compressive at points on the lower surface of the notch-tip to tensile at points on the upper surface of the notch-tip.

Fig. 6a illustrates the time history of the evolution of the effective plastic strain at the point where the ductile failure occurs. The rate of increase of the effective plastic strain or equivalently the effective plastic strain-rate has the highest value for a/b = 0.4. Approximate values of the effective plastic strain-rate at these four points soon after the loading wave arrives at the notch-tip equal 1.0×10^5 , 6×10^4 , 1.5×10^4 and 1×10^4 /s ,respectively, for a/b = 0.4, 1.0, 2.0 and 10.0. Thus typical peak values of the effective plastic strain-rate are in the range of 10^4 to 10^5 /s for the four values of a/b, and the lower value of the effective plastic strain rate occurs for a larger value of a/b. The time history of the evolution of the maximum principal tensile stress at the point where the brittle failure ensues is shown in Fig. 6b. The angular locations, in the deformed configuration and measured counter-clockwise, of these points are 80, 50, 30 and 15° , respectively, for a/b = 0.4, 1.0, 2.0 and 10.0.



Fig. 4. Deformed shapes of the four notch-tips, and the locations in the deformed configuration of points from where the brittle and the ductile failures initiate.



Fig. 4. (Continued)



Fig. 5. For the impact speed of 30 m/s, the angular variation of (a) the effective plastic strain, and (b) the normalized maximum principal stress at the centroids of elements abutting the notch-tip. Results are plotted at the instant of the initiation of the brittle failure, and the angular locations are in the reference configuration.



Fig. 6. For the four elliptic notch-tips, time histories of (a) the effective plastic strain, and (b) the normalized maximum principal stress at the points where at the instant of the initiation of the brittle failure maximum values of the effective plastic strain and the maximum principal stress occur, respectively.

From the plots given in Fig. 7 of the impact speed vs. the times of initiation of the brittle and the ductile failures, the impact speeds at which the failure mode transitions from the brittle to the ductile are found to be 19.0 and 19.3 m/s, respectively, for a/b=0.4 and 1.0; for a/b=2.0 and 10.0, the ductile failure preceded the brittle failure for all impact speeds considered herein. It suggests that out of the four notch shapes scrutinized, the failure mode transition occurs only for elliptic notches with a/b = 0.4 and 1.0. It may provide an explanation for the differences in the test observations of Kalthoff and Zhou et al. who seem to have tested prenotched plates made of very similar materials. Because of the different techniques used to cut notches, the shapes of the notch-tips may not have been identical in the specimens used in the two laboratories. Whereas the time of initiation of the ductile failure for a circular notch-tip decreases noticeably from 27 μ s at $v_0 = 15$ m/s to 15 μ s for $v_0 = 35$ m/s, it decreases from 15 μ s at $v_0 = 15$ m/s to 13 μ s at $v_0 = 35$ m/s for an elliptic notch tip with a/b = 10. For a very blunt notch-tip with a/b = 0.4, the two failure modes initiate almost simultaneously, but that is not the case for a/b = 1.0, 2.0 and 10.0. For each one of the four values of a/b, the times of initiation of both the brittle and the ductile failures decrease with an increase in the impact speed; the slopes of these curves strongly depend upon the value of a/b.

Batra and Jaber (2001a) have investigated the effect of the radius r_0 of the circular notch-tip on the failure mode transition speed v_{cr} . They found that for the mild steel being studied,

$$v_{\rm cr} = 1179 r_0^{0.4655} \,\,{\rm m/s} \tag{19}$$

where r_0 is in meters, which compares well with the relation

$$v_{\rm cr} = 1350 r_0^{0.5} \,\,{\rm m/s} \tag{20}$$

obtained by Kalthoff (2000) from his experimental findings. Eqs. (19) and (20) were derived from data obtained with values of r_0 in different ranges, e.g. for Eq. (19) 0.05 mm $\leq r_0 \leq 0.3$ mm, and for Eq. (20) 0.25 mm $\leq r_0 \leq 0.85$ mm.

Ching and Batra (2001a,b) analyzed the corresponding linear elastostatic problem in which the essential boundary condition $(17)_1$ was replaced by a pressure loading on the impacted surface, and the four corners of the plate were rigidly clamped. They found that the mode-I and the mode-II stress intensity factors were the same for the four shapes of the notch-tip. However, the shape of the notch-tip strongly influenced the angular variations of the maximum principal stress and the maximum shear stress at points on the surface of the notch-tip. Also, the peak values of these stresses depended upon the shape of the notch-tip.

Batra and Jaber (2001b) have shown, through numerical experiments, that the height 75 mm of the specimen above and below the two notches in Fig. 1 can be reduced to 50 mm without noticeably affecting the times of initiation of the brittle and the ductile failures.



Fig. 7. For the four elliptic notch-tips, the dependence upon the impact speed v_0 of the times of initiation of the brittle and the ductile failures.



Fig. 7. (Continued)

3.2. Effect of a circular hole in front of a circular notch-tip

We now investigate the effect on the thermomechanical deformations of the prenotched plate of a circular hole of radius $r_0 = b$ situated ahead of the circular notchtip of radius $r_0 = 0.15$ mm. We first consider the case when the center of the circular hole is on the axis of the notch with the distance between the centers of the circular hole and the circular notch-tip equal to d, and subsequently when the center of the hole is at a distance e from the axis of the notch. The two configurations are depicted in Fig. 1b and c, respectively. Even though many of the structural materials are porous, we note that the configurations depicted in Fig. 1a and b have not been tested experimentally.

3.2.1. Circular hole located on the axis of the notch

Fig. 8 shows the dependence of the times of initiation of the brittle and the ductile failures upon the impact speed for $d/r_0 = 3, 4, 5$ and 6. In each case the brittle failure precedes the ductile failure for impact speeds of 10, 15, 20, 25, 30 and 35 m/s. For an impact speed of 35 m/s, the brittle failure ensues at a point, F, on the lower surface of the circular hole that is closer to the circular notch-tip. The line joining P to the center of the hole makes an angle in the deformed configuration of 9° counterclockwise from the axis of the notch for $d=3r_0$ and 18° for the other three values of d. For $d/r_0 = 3$, the time of initiation of the brittle failure decreased from 12.75 to 10.92 µs as the impact speed was increased from 10 to 35 m/s; however, the corresponding change in the time of initiation of the brittle failure is from 20.46 to $13.08 \,\mu s$ for $d=6r_0$. In the absence of the circular hole, the time of initiation of the brittle failure equalled 26.4 and 18 µs for $v_0 = 10$ and 35 m/s, respectively. Thus the presence of a hole even relatively far away from the notch-tip influences both the time of initiation of the brittle failure and the point where it initiates. At the impact speed of 35 m/s, the time of initiation of the ductile failure is relatively unaffected by the presence and the location of the circular hole. For $d = 6r_0$, the ductile failure initiates from a point Q on the surface of the notch tip; the line joining Q to the center of the notch tip makes an angle in the current configuration of 27° clockwise with the axis of the notch for $v_0 = 10$, 15 and 20 m/s but 45° for $v_0 = 25$, 30 and 35 m/s. For $d/r_0 = 3$, 4 and 5, the location of the point of initiation of the ductile failure varied with the impact speed. For $v_0 = 35$ m/s, the ductile failure initiated at t = 13.41 and 15 µs, respectively, in the presence of the circular hole with $d=6r_0$ and in the absence of the hole.

We have plotted in Fig. 9 the deformed shapes of the notch tip and the circular hole for $v_0 = 30$ m/s and at the instant of the initiation of the ductile failure. The locations in the deformed configurations of the points of initiation of the two failure modes are also depicted. An interesting observation is that the circular hole is also severely distorted and the brittle failure initiates from a point near the surface of the circular hole rather than from a point close to the surface of the notch-tip. Except for $d/r_0 = 3$, the ductile failure ensues from a point adjacent to the surface of the notch-tip.

For an impact speed of 25 m/s, Fig. 10a-c evinces the fringe plots of the effective plastic strain, the normalized maximum principal stress and the nondimensional

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Fig. 8. For the four locations of the circular hole ahead of the notch-tip, the dependence upon the impact speed of the times of initiation of the brittle and the ductile failures.



Fig. 8. (Continued)



Fig. 9. Deformed shapes of the notch-tip and the circular hole, and the locations of the points of initiation of the brittle and the ductile failures.



Fig. 9. (Continued)



Fig. 10. For the four locations of the circular hole ahead of the notch-tip and the impact speed of 25 m/s, fringe plots in a small region around the notch-tip and the circular hole of (a) the effective plastic strain, (b) the shear stress σ_{12} , and (c) the maximum principal stress. The times at which these results are plotted equal 17.2, 21.6, 23.6 and 22.1 µs, respectively, for $d/r_0 = 3$, 4, 5 and 6.





Fig. 10. (Continued)





Fig. 10. (Continued)





Fig. 10. (Continued)





Fig. 10. (Continued)





Fig. 10. (Continued)

shear stress α_{12} in a small region including the notch-tip and the circular hole. In each case, results are plotted at the instant the ductile failure ensues since it occurs later; the times when these figures are plotted equal 14.0, 15.0, 17.3 and 18.7 µs for $d/r_0 = 3, 4, 5$ and 6, respectively. Except for the values of the time of initiation of the failure, qualitatively similar results were obtained for impact speeds of 10, 15, 20, 30 and 35 m/s. In the absence of the circular hole, the brittle and the ductile failures initiate respectively at 22 and 20 µs after impact. Thus, the presence of the circular hole at $d/r_0 = 3$, 4, 5 or 6 makes the brittle failure occur sooner. The fringes of the effective plastic strain plotted in Fig. 10a indicate four severely deformed regionstwo emanating from a point on the lower surface of the notch-tip and the other two from a point on the lower surface of the circular hole. Note that the lower surface of the notch-tip is closer to the impacted edge. The qualitative nature of results is the same for each one of the four values of d. The most intensely deformed narrow region is around the line joining a point on the lower surface of the notch-tip to a point on the upper surface of the hole. The locations, in the reference configuration, of these points can be discerned from the plots, given in Fig. 11a of the distribution of the effective plastic strain in elements adjoining the circular notch-tip and the hole. With an increase in the value of d/r_0 from 3 to 6, the angular position in the deformed configuration of the point on the surface of the notch-tip where the effective plastic strain is maximum shifts from -16 to -27° , and that on the hole from 154 to 162° ; in each case the positive angles are measured clockwise from the axis of the notch. Without the hole, the angular position in the deformed configuration of the point on the surface of the circular notch-tip where the effective plastic strain is maximum equals -50° . Thus the presence of the hole shifts the location of the nucleus of the shear band on the notch-tip surface towards the axis of the notch. Within the intensely deformed narrow region joining a point on the circular notch-tip to a point on the circular hole, the maximum value of the effective plastic strain at a point near the circular hole is more than that at a point adjacent to the circular notch-tip. A small region around the point located, in the reference configuration, at -70° on the surface of the notch-tip is also intensely deformed. With an increase in the value of d/r_0 from 3 to 6, the maximum value of the effective plastic strain in this region increases from 0.07 to 0.2 and that in the material around the point P decreases from 0.27 to about 0.25; the point P is located, in the reference configuration, on the surface of the notch tip at an angular position of about -30° . As is evident from the results plotted in Fig. 11a, a small region surrounding point T, located in the reference configuration at about -60° , on the surface of the circular hole is also severely deformed; the maximum effective plastic strain at a point in this region equals approximately 0.20 for $d/r_0 = 3$ and 6. The corresponding values of the effective plastic strain in the intensely deformed region around the point R, located in the reference configuration at about 150° , on the circular hole are 0.37 and 0.28. Thus there are four regions wherein the effective plastic strain is large; two of these regions are around the points on the surface of the notchtip and the other two are around the points R and T on the surface of the circular hole. Fringe plots of the maximum principal stress and the shear stress σ_{12} given respectively in Fig. 10b and c reveal that the maximum principal stress is compressive and the magnitude of the shear stress is large in intensely deformed regions.



Fig. 11. For the impact speed of 25 m/s, the angular variation of (a) the effective plastic strain, and (b) the normalized maximum principal stress at the centroids of elements abutting the notch-tip and the circular hole. Results are plotted at the instant of the initiation of the ductile failure, and angular positions are in the reference configuration.



Fig. 11. (Continued)



Fig. 12. For the impact speed of 25 m/s, the time history of the effective plastic strain and the normalized maximum principal stress at the points (a) near the notch-tip and (b) near the hole where their maximum values occur at the instant of the initiation of the ductile failure.



Fig. 12. (Continued)

For an impact speed of 25 m/s, Fig. 12a shows the time histories of the evolution of the effective plastic strain and the maximum principal stress at points close to the surface of the notch-tip where their maximum values occur at the instant of the initiation of the ductile failure; the corresponding results for points around the circular hole are depicted in Fig. 12b. A comparison of the evolution of the effective plastic strain at the points near the notch-tip and the circular hole suggests that, for $d/r_0 = 0.3$, the initial effective plastic strain-rates at the former and the latter locations are about 6.8×10^4 /s and 8×10^4 /s, respectively. At points around the notch tip and the circular hole, the initial effective plastic strain-rate decreases with an increase in the value of d/r_0 . In each one of the four plots of Fig. 12, the effect of the circular hole is more noticeable for $d/r_0 = 3$ than for its other three locations.

3.2.2. Circular hole located away from the axis of the notch

For an impact speed of 35 m/s and $e/r_0 = \pm 1$ and ± 3 , we have plotted in Fig. 13a–c fringes of the effective plastic strain, the normalized shear stress σ_{12} and the normalized maximum principal stress. These results are plotted at t = 15.3, 15.6, 13.1and 17.0 μ s, respectively, for $e/r_0 = 1$, 3, -1 and -3; these times correspond to the times of initiation of the brittle failure for $e/r_0 = -1$ and -3, and to the time of onset of the ductile failure for $e/r_0 = 1$ and 3. Positive values of e/r_0 indicate that the hole is located above the axis of the notch (cf. Fig. 1c). The angular distributions of the effective plastic strain and the normalized maximum principal stress at points abutting the circular notch tip and the circular hole are exhibited in Fig. 14 wherein the angular positions of points are in the reference configuration. For $e/r_0=3$, the presence of the hole has a modest effect on the location of the point on the notch surface where the maximum effective plastic strain occurs, and on the location and shape of the severely deformed region. The angular distribution of the effective plastic strain around the surface of the notch-tip and its maximum value are quite similar to those computed for the circular notch-tip in the absence of the circular hole and depicted in Fig. 4a. Note that the impact speeds and the times when these results are plotted are different in the two cases. Whereas the circular hole with its center located at $e/r_0 = 3$ does not influence much the deformations of the material around the notch-tip, the hole located at the other three locations significantly affects deformations of the material around the notch-tip and the hole. For $e/r_0=1$, a large portion of the lower surface of the notch-tip $(-90^\circ \leq \pi \leq 0^\circ)$ is severely deformed, and there is also a small region of the upper surface where the effective plastic strain is noticeable. In the absence of the hole, the effective plastic strain at points on the upper surface of the notch-tip is quite small. There appear to be four regions of intense plastic deformations; two emanating from a point on the right side of the lower surface of the hole, one within the small region joining a point on the upper surface of the notch-tip to a point on the upper surface of the circular hole, and the fourth one originating from a point on the lower surface of the notch-tip and propagating at an angle of approximately -135° to the notch-axis. For $e/r_0 = -1$ and -3, the intense plastic deformations originating from a point on the bottom right side of the circular hole propagate essentially horizontally. A narrow region joining a point on the lower surface of the notch-tip to a point on the upper surface



Fig. 13. For the impact speed of 35 m/s, fringe plots of (a) the effective plastic strain, (b) the normalized shear stress σ_{12} , and (c) the normalized maximum principal stress in a small region enclosing the notch-tip and the circular hole. Times at which results are plotted equal, respectively, 15.3, 15.6, 13.1 and 17.0 µs for $e/r_0 = 1$, 3, -1 and -3.





Fig. 13. (Continued)





Fig. 13. (Continued)





Fig. 13. (Continued)





Fig. 13. (Continued)







Fig. 14. For the impact speed of 35 m/s, the angular variations of the effective plastic strain, and the normalized maximum principal stress at the centroids of elements adjoining the surface of (a) the notchtip and (b) the circular holes; the angular locations are in the reference configuration.



Fig. 14. (Continued)

of the circular hole is very intensely deformed. For the case of the $e/r_0 = -3$, the effective plastic strain in the material above this narrow region is noticeably more than that in the region below it. For $e/r_0 = -1$ and -3, there is a small leaflet like region pointing in the -135° direction wherein the effective plastic strain is also large; this region surrounds points on the lower surface of the notch-tip.

Fringe plots of the shear stress σ_{12} depicted in Fig. 13b suggest that for $e/r_0 = -1$ and -3, the magnitude of the shear stress is the largest in narrow essentially horizontal regions originating from a point on the lower right surface of the circular hole. The magnitude of the shear stress is also quite large in the other narrow regions joining a point on the lower surface of the notch-tip to a point on the upper surface of the circular hole. For all four locations of the circular hole, the magnitude of the shear stress is large in regions which have significant plastic strains.

Fringe plots of the maximum principal stress and its distribution around the surfaces of the notch-tip and the circular hole reveal that the presence of the hole at the four locations does not appreciably influence the distribution, exhibited in Fig. 14, of the maximum principal stress around the surface of the notch-tip. However, for $e/r_0 = 1$ and 3, the maximum principal stress at points on the lower left side of the circular hole is significantly larger than that at points similarly situated for $e/r_0 = -1$ and -3. For all four locations of the circular hole, the distribution of the maximum principal stress at points on the upper surface of the circular hole is qualitatively similar except that for $e/r_0 = -3$, the region around the circular hole with large values of the maximum tensile principal stress is quite small.

We have plotted in Fig. 15 the deformed shapes of the notch-tip and the circular voids for the four locations of the hole, and have also marked the locations of the points from where the brittle and the ductile failures initiate. In each case, the circular hole is displaced in the direction of impact, and the horizontal movement is largest for $e/r_0 = -3$ because it is then directly ahead of the impacted edge. The brittle failure initiates from a point on the surface of the circular hole for all four locations of the hole. For $e/r_0 = +1$, the point of initiation of the ductile failure is also adjacent to the surface of the hole, but for $e/r_0 = +3$, the ductile failure ensues from a point near the lower surface of the notch-tip which is the case when there is no hole. Thus, the presence of the holes with $e/r_0 = \pm 3$ influences more the location of the brittle failure.

Fig. 16 depicts the time histories of the effective plastic strain and the maximum principal stress at points adjacent to the surfaces of the notch-tip and the circular hole where their maximum values occur at the time of initiation of the ductile failure. For $e/r_0 = -1$ and -3, the initial effective plastic strain at the points near the notch-tip surface and the circular hole is 1×10^5 /s which is a little higher than that when there is no hole or when the hole is located above the axis of the notch. The dependence upon the impact speed of the times of initiation of the brittle and the ductile failures is exhibited in Fig. 17. Comparing these with the results plotted in Fig. 7 we see that the presence of the hole at $e/r_0 = \pm 1$ and 3 accelerates the initiation of the brittle failure while that at $e/r_0 = -3$ retards it. However, the holes located at $e/r_0 = -1$ and -3 enhance the onset of the ductile failure.



Fig. 15. Deformed shapes of the notch-tip and the circular hole, and the locations of the points of initiation of the brittle and the ductile failures.



Fig. 15. (Continued)



Fig. 16. For the impacts speed of 35 m/s, time histories of the effective plastic strain and the maximum principal stress at points near the surface of (a) the notch-tip and (b) the circular hole where their maximum values occur at the time of initiation of the ductile failure.



Fig. 16. (Continued)



Fig. 17. For the four locations of the circular hole with its center either above or below the axis of the notch, the dependence upon the impact speed of the times of initiation of the brittle and the ductile failures.



Fig. 17. (Continued)

4. Conclusions

We have numerically studied transient plane strain thermomechanical deformations of a prenotched plate with the edge surface between the two notches subjected to an impact load. The impact load is simulated by prescribing null tangential tractions and a time dependent normal velocity at the impacted plate points. The material response of the plate is modeled by the Bodner–Partom relation. The effects of the shape of the notch-tip and of the presence of a circular hole located ahead of the circular notch-tip on the initiation of a failure mode are also scrutinized.

It is found that for very blunt and circular notch-tips the failure mode transitions from the brittle to the ductile with an increase in the impact speed. However, for two very sharp elliptic notch tips, the ductile failure precedes the brittle failure for each one of the six impact speeds studied. Points on the surface of the notch-tip where the ductile failure initiates move towards the axis of the notch-tip with an increase in the ratio of the length of the principal axis of the elliptic notch-tip that is along the axis of the notch to the one perpendicular to it. Typical values of the effective plastic strain rate are 5×10^4 /s and the temperature rise at the instant of the initiation of the shear band is 120 °C.

When a circular hole of radius r_0 is located directly ahead of the circular notch-tip of radius r_0 , the brittle failure always preceded the ductile failure and the former initiated from a point on the lower surface of the circular hole. Except when the distance between the centers of the circular hole and the circular notch-tip equals $3r_0$, the ductile failure initiated from a point on the lower surface of the circular notch-tip. The circular hole is noticeably displaced in the direction of impact and is also severely deformed. The largest plastic deformations occur in the region surrounding the line joining a point on the lower surface of the notch-tip to a point on the upper surface of the circular hole.

We have also scrutinized the thermomechanical deformations of the prenotched plate when the circular hole is located either above or below the axis of the notch. The presence of the hole below the axis of the notch enhances the initial effective plastic strain-rate at a point on the lower surface of the notch-tip and accelerates the initiation of the brittle failure; the brittle failure ensues from a point on the surface of the circular hole. When the center of the hole is located at a vertical distance equal to the radius of the hole either above or below the axis of the notch, both the brittle and the ductile failures originate from points adjacent to the surface of the circular hole. However, when this vertical distance equals 3 times the radius of the hole, the points of initiation of the brittle failures are near the surface of the hole but those of the ductile failure close to the lower surface of the notchtip.

The presence of the hole ahead of the notch-tip does not affect the evolution of porosity at points adjacent to either the notch-tip or the circular void. This is primarily because our computations were stopped soon after a failure initiated. Computations could not be carried into the post-failure regime because of severe distortions of the mesh.

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