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On Non-Classical Boundary Conditions

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1. Introduction

To determine the deformation of a body from the system of differential equations governing its thermo-mechanical deformations, we need side conditions such as boundary conditions. The most frequently employed boundary conditions are those of place and/or traction in a purely mechanical problem and of temperature and/or heat flux in a thermal problem. These and other more familiar mechanical boundary conditions are summarized by SEWELL [1, eqns. (9)–(13)]. The frequently considered thermal boundary conditions are discussed by CARSLAW & JAEGER [2, Chapter 1]. It seems that the interest in non-classical mechanical boundary conditions originated in the study of elastic stability, and a convenient reference for the various mechanical loadings considered up to 1961 is the book by BOLOTIN [3]. In this book, BOLOTIN discusses many elastic-stability problems under *follower loads*, defined as those which follow in a prescribed manner the deformation of the surface element upon which they are acting. More general than these are the *configuration-dependent loadings* [1] defined* by an *a priori* assumption that the load acting at a material boundary point is calculable from an assigned function of at most the displacement of that point and its neighbors. SEWELL also gives sufficient conditions for such loadings to be conservative. In [4], NEMAT-NASSER states that surface tractions are usually prescribed so as to represent the interaction between a deforming body and its surroundings. He proposes that applied surface tractions on a body be represented by forces which are defined on material neighbourhoods as vector valued functions of the displacement, velocity, acceleration and their first and higher order gradients with respect to the particle positions in

* *Cf.* SEWELL [1, p. 327].

a reference configuration and calls these *motion-dependent* loads. In the specification of these motion-dependent loads, it is tacitly assumed that the body interacts locally with the environment. In a general sense, the configuration-dependent loadings and the motion-dependent loadings follow the material in that the load senses the deformation in the neighborhood of its point of application and varies with it.

Here, I also adopt the viewpoint that the boundary conditions describe how the body interacts with its environment, which I call a *loading device* in the following discussion. It is assumed that the loading device L can be regarded as a deformable continuum. Presuming a rather simple theory for it, I strive to derive an approximate theory with the hope that the deduced boundary conditions account somewhat for the deformation of L. Said differently, an attempt is made to express the balance laws for L in terms of quantities defined on the common interface C and then, assuming the contact conditions at C, to derive the boundary conditions. In this context it may be mentioned that the prescription of boundary conditions can alternatively be regarded as specification of the constitutive equations for L and the contact conditions at the interface C. For example, assigning zero surface displacements at the bounding surface of a body B is equivalent to saying that B is permanently glued to L, the loading device is a rigid body, and L cannot suffer any motion. As should become clear after the reader has gone through the following discussion, the proposed technique of deriving balance laws for L and thence the boundary conditions works at most for the class of loading devices for which C always consists of the same material particles of L.

What emerges is that the body and the loading device interact nonlocally in the sense that, in a purely mechanical problem, the surface tractions at a point of C depend upon the deformation of all points of C and not just on the deformation of its neighborhood. An analogous result holds for the thermal boundary condition. These results are in sharp contrast to the prescription of forces and heat flux now in use. Also, this study reveals that the surface tractions at C can depend on the field of displacement defined on C and *not* on the inward normal derivatives of displacement. A similar result holds for the heat flux.

2. Derivation of Balance Laws for Loading Devices

The technique pursued here to deduce the balance laws for the loading device is the one customarily used in deriving the balance laws for a shell or a rod* from the three dimensional equations. The details are slightly different since, in the case of both shells and rods, advantage is usually taken of the special geometry of the body. Though the procedure followed below would work for more general continuum theories** for L, for the sake of simplicity I start with a rather conventional (non-polar) continuum theory. In particular, it is assumed that the loading device obeys the following balance laws of mass, linear momentum, moment of momentum and energy and an entropy inequality $(2.1)_5$. In Cartesian

* For a detailed discussion of these, see the recent articles by NAGHDI [5] and ANTMAN [6].

** *E.g.*, those which account for director stresses, body couples, couple and multipolar stresses, *etc.*

tensor notation, these laws take the forms

$$
\begin{aligned}
&\frac{d}{dt}\int \rho^* \, dV = 0,\\
&\frac{d}{dt}\int \rho^* \dot{x}_i \, dV - \int T^*_{ia} \, dS_a - \int b^*_i \, dV = 0,\\
&\frac{d}{dt}\int \rho^* x_{[i} \dot{x}_{j]} \, dV - \int x_{[i} T^*_{j]a} \, dS_a - \int x_{[i} b^*_{j]} \, dV = 0,\\
&\frac{d}{dt}\int \left(\varepsilon^* + \frac{\rho^*}{2} \dot{x}_i \dot{x}_i\right) dV - \int (\dot{x}_i T^*_{ia} - Q^*_a) \, dS_a - \int (b^*_i \dot{x}_i + \gamma^*) \, dV = 0,\\
&\frac{d}{dt}\int \eta^* \, dV + \int J^*_a \, dS_a - \int h^* \, dV \geqq 0.
\end{aligned}
\tag{2.1}
$$

The integrands are considered as functions of time t and the material co-ordinates X_a, interpretable as co-ordinates of a particle in a convenient reference configuration. For a given loading device, the reference configuration is a region R, independent of time, the region of integration in (2.1). On its boundary ∂R, dS_a denotes the outward directed vector element of area. Further, ρ^* is the mass, b^*_i the body force, γ^* the supply of internal energy, ε^* the internal energy, η^* the entropy and h^* the entropy supply, each measured per unit reference volume. In $(2.1)_4$ and $(2.1)_5$, Q^*_a and J^*_a are the heat flux vector and the entropy flux vector reckoned per unit area in the reference configuration. The tensor T_{ia} is the Piola-Kirchhoff stress tensor, the vector x_i denotes the present coordinates of a particle and $\dot{x}_i$ denotes its velocity. If, in $(2.1)_5$, one sets

$$J^*_a = \frac{Q^*_a}{\theta}$$

and

$$h^* = \frac{\gamma^*}{\theta},$$

where θ, the temperature of a particle, is assumed to be strictly positive, one recovers the Clausius-Duhem inequality. To obtain MÜLLER's * entropy inequality for a supply free loading device, one needs to take $h^* = 0$. In (2.1), line or surface concentrations of mass, energy and entropy, *etc.* are assumed to be absent, and it is also presumed that there is no production of mass, linear momentum, moment of momentum and energy within L, whereas $(2.1)_5$ asserts that the production of entropy is non-negative. Also under the change of frame

$$
\begin{aligned}
&\hat{x}_i(X_a, t) = Q_{ij}(t)\, x_j(X_a, t) + c_i(t),\\
&Q_{ij}(t)\, Q_{kj}(t) = \delta_{ik}, \qquad \det[Q_{ij}(t)] = \pm 1
\end{aligned}
\tag{2.2}
$$

* See, *e.g.*, MÜLLER [7].

where δ_{ij} is the Kronecker delta; it is conventional to assume that the above quantities transform as follows:

$$\begin{gathered}\hat{\rho}^*=\rho^*, \quad \hat{\varepsilon}^*=\varepsilon^*, \quad \hat{\theta}=\theta, \\ \hat{\eta}^*=\eta^*, \quad \hat{h}^*=h^*, \quad \hat{\gamma}^*=\gamma^*, \\ \hat{Q}_a^*=Q_a^*, \quad \hat{J}_a^*=J_a^*, \\ \hat{T}_{ia}^*=Q_{ij}T_{ja}^*, \quad \hat{b}_i^*-\hat{\rho}^*\ddot{\hat{x}}_i=Q_{ij}(b_j^*-\rho^*\ddot{x}_j).\end{gathered} \tag{2.3}$$

In preparation for the discussion to follow, it is helpful to observe that the integral equations (2.1) are of the form

$$\frac{d}{dt}\int \psi^*\,dV-\int \mathfrak{A}_a^*\,dS_a-\int \sigma^*\,dV=\int P^*\,dV. \tag{2.4}$$

Here, ψ^*, σ^* and P^* are tensor fields of the same order defined over R and $\mathfrak{A}_a^*$ is a tensor field of order greater by one than that of ψ^* and σ^*. We interpret P^* as the production of ψ^* within R and the second and third terms on the left-hand side of (2.4) as the rate of increase of ψ^* because of inflow through the boundary of R and growth at places within R. It is usual to call $\mathfrak{A}^*$ the flux and σ^* the supply of ψ^*. To express (2.4) in terms of quantities defined on the part C of the boundary of the loading device, it is convenient to introduce a curvilinear co-ordinate system defined by the transformation

$$Z^a=Z^a(X_b) \qquad (a,b=1,2,3),$$

such that

(i) the common interface C is contained in the co-ordinate surface $Z^3=0$,

(ii) the co-ordinate curves

$$Z^1=\text{const.}, \qquad Z^2=\text{const.}$$

intersect F defined by

$$F\equiv\partial R-C$$

in one point only,

(iii) Z^a are of class C^1, and

(iv) $J^{-1}\equiv\det\left[\dfrac{\partial Z^a}{\partial X_b}\right]\neq 0$ everywhere in the region V defined by

$$V\equiv\{Z^a\colon\ Z^a\in R,\ (Z^\alpha,0)\in C\}. \tag{2.7}$$

It may be remarked that the stated conditions on the co-ordinate system Z are more restrictive than is necessary for our purpose. These conditions on the co-ordinate system Z do not require that the Z^3 co-ordinate curves be along the normal to the surface $Z^3=0$. Hereafter, we shall denote the material co-ordinates of a point on C by Z^α $(\alpha=1,2)$.

Now consider a material tube T, $T\subset R$, with one end consisting of points of $\sigma\subset C$ and the lateral surface ∂T_0 consisting of points of the set

$$\{Z^a\colon Z^a\in R,\ (Z^\alpha,0)\in\partial\sigma\}.$$

With the definition

$$\sigma_0\equiv T\cap F,$$

we have

$$\partial T = \sigma \cup \partial T_0 \cup \sigma_0.$$

Let a parametric representation of $\partial\sigma \subset C$ and of σ_0 be*

$$\begin{aligned} Z^\alpha &= Z^\alpha(s), \\ X^a &= \hat{X}^a(Z^1, Z^2). \end{aligned} \tag{2.8}$$

I now introduce the following definitions:

$$\begin{aligned} \psi^+(Z^a) &\equiv \psi^*(Z^a)\,J(Z^a), \\ \vec{\sigma}^+(Z^a) &\equiv \vec{\sigma}^*(Z^a)\,J(Z^a), \\ P^+(Z^a) &\equiv P^*(Z^a)\,J(Z^a). \end{aligned} \tag{2.9}$$

Whereas ψ^* is the density per unit Euclidean volume, ψ^+ is the density per unit *co-ordinate volume* $dZ^1\,dZ^2\,dZ^3$. We now try to express the flux $\mathfrak{h}^*_a$ in terms of the flux $\mathfrak{h}^+_a$ measured per unit co-ordinate area. Note that

$$\begin{aligned} \mathfrak{h}^a\,dS_a &= \mathfrak{h}^a\,\varepsilon_{abc}\frac{\partial X^b}{\partial Z^g}\frac{\partial X^c}{\partial Z^k}\,dZ^g\,dZ^k, \\ &= \mathfrak{h}^a J\frac{\partial Z^f}{\partial X^a}\,\varepsilon_{fgk}\,dZ^g\,dZ^k, \\ &= \mathfrak{h}^{+f}\,\varepsilon_{fgk}\,dZ^g\,dZ^k, \end{aligned}$$

where we have set

$$\mathfrak{h}^{+f} \equiv \mathfrak{h}^a J\frac{\partial Z^f}{\partial X^a}.$$

In (2.10), ε_{abc} is the alternating tensor having values 1 or -1 according as a, b, c form an even or odd permutation of 1, 2, 3 and vanishing otherwise. Because of $(2.8)_1$ and our choice of the material tube T, on the mantle ∂T_0 of T,

$$\begin{aligned} \varepsilon_{abc}\,dZ^b\,dZ^c &= \varepsilon_{a\alpha 3}\frac{dZ^\alpha}{ds}\,ds\,dZ^3, \\ &= \nu_a\,ds\,dZ^3, \end{aligned}$$

where

$$\nu_a(Z^\beta) \equiv \varepsilon_{a\alpha 3}\frac{dZ^\alpha}{ds},$$

points along that outer normal to $\partial\sigma$ which lies in the surface $Z^3 = \text{const.}$ We note that ν_a does not depend upon Z^3. Thus

$$\int_{\partial T_0} \mathfrak{h}^{*a}\,dS_a = \int_{\partial T_0} \nu_a\,\mathfrak{h}^{+a}\,ds\,dZ^3 = \int_{\partial T_0} \nu_\alpha\,\mathfrak{h}^{+\alpha}\,ds\,dZ^3.$$

---

* Let σ_0 be given by

$$Z^3 = Z^3(Z^1, Z^2).$$

Then on σ_0,

$$X^a = X^a(Z^1, Z^2, Z^3(Z^1, Z^2)) \equiv \hat{X}^a(Z^1, Z^2).$$

where the second equality follows from the fact that $\nu_3=0$. Using the definitions

$$
\begin{aligned}
&z(Z_0^\alpha)\equiv\{Z^3:(Z_0^\alpha, Z^3)\in F\},\\
&P^\psi(Z^\alpha,0)\equiv\varepsilon_{abc}\left[\mathfrak{h}_a^*\frac{\partial X^b}{\partial Z^1}\frac{\partial X^c}{\partial Z^2}\bigg|_{Z^3=0}+\mathfrak{h}_a^*\frac{\partial \hat{X}^b}{\partial Z^1}\frac{\partial \hat{X}^c}{\partial Z^2}\bigg|_{Z^3=z}\right],\quad Z^\alpha\in\sigma,
\end{aligned}
\tag{2.13}
$$

(2.9) and (2.12), we can write (2.4) in the form

$$
\begin{aligned}
&\frac{d}{dt}\int_T\psi^+\,dZ^1\,dZ^2\,dZ^3-\int_\sigma P^\psi\,dZ^1\,dZ^2-\int_{\partial T_0}\mathfrak{h}_\alpha^+\,\nu^\alpha\,ds\,dZ^3\\
&\qquad-\int_T\vec{\sigma}^{\,+}\,dZ^1\,dZ^2\,dZ^3=\int_T P^+\,dZ^1\,dZ^2\,dZ^3.
\end{aligned}
\tag{2.14}
$$

We can interpret $P^\psi(Z_0^\alpha,0)$ defined by $(2.13)_2$ as the total flux of ψ through the ends of the curve

$$
\{Z^\alpha:\ Z^\alpha\in R,\ Z^\alpha=Z_0^\alpha\}.
\tag{2.15}
$$

With ψ^+ and $\mathfrak{h}^+$, *etc.*, there is the advantage that we can avoid introducing co-variant differentiation. Since I am using the material description throughout this paper, the operators $\frac{d}{dt}$ in (2.14) and the superposed dot used below are equivalent to partial differentiation with respect to time t. We now introduce the following definitions and thus express (2.14) in terms of quantities defined on C:

$$
\begin{aligned}
\psi(Z^\alpha,0)&\equiv\int_0^z\psi^+(Z^a)\,dZ^3,\\
\vec{\sigma}(Z^\alpha,0)&\equiv\int_0^z\vec{\sigma}^{\,+}(Z^a)\,dZ^3,\\
\mathfrak{h}_b(Z^\alpha,0)&\equiv\int_0^z\mathfrak{h}_b^+(Z^a)\,dZ^3,\\
P(Z^\alpha,0)&\equiv\int_0^z P^+(Z^a)\,dZ^3.
\end{aligned}
\tag{2.16}
$$

Recalling the definition $(2.13)_1$ of z, we see that the integration in (2.16) is along the curve parallel to the Z^3 co-ordinate curve and having $(Z^\alpha,0)$, $(Z^\alpha, z(Z^\alpha))$ as the terminal points. In terms of the definitions (2.16), we rewrite (2.14) as

$$
\frac{d}{dt}\int_\sigma\psi\,dZ^1\,dZ^2-\int_\sigma P^\psi\,dZ^1\,dZ^2-\oint_{\partial\sigma}\mathfrak{h}_\alpha\nu^\alpha\,ds-\int_\sigma\vec{\sigma}\,dZ^1\,dZ^2=\int_\sigma P\,dZ^1\,dZ^2.
\tag{2.17}
$$

The second term in (2.17) can be viewed as the supply of ψ from within the loading device or as the production of ψ in the material surface element σ. Thus even if we assume that there is no production of ψ within L, *i.e.* $P^*=0$ in (2.4) and hence $P=0$ in (2.17), there may be a finite contribution to ψ due to this term. It is clear from (2.9), (2.11), (2.13) and (2.16) that the values of ψ, $\mathfrak{h}$, $\vec{\sigma}$, *etc.* depend upon the geometry of the loading device and on the choice of co-ordinate system Z. With the postulate that (2.17) holds for every surface element of C and the assumption that the integrands are continuous, (2.17) is equivalent to the local equation

$$
\dot{\psi}-P^\psi-\mathfrak{h}^\alpha{}_{,\alpha}-\vec{\sigma}=P.
\tag{2.18}
$$

To obtain (2.18), the surface divergence theorem has been used. Here and below, a comma stands for partial differentiation with respect to Z^α, that is

$$\mathcal{T}^\alpha{}_{,\alpha} = \frac{\partial \mathcal{T}^\alpha}{\partial Z^\alpha}.$$

Keeping in mind the derivation of (2.18) from (2.4), we have the balance laws (2.1) which take the following forms in terms of quantities defined on C:

$$\begin{aligned}
&\dot{\rho} = 0,\\
&\dot{l}_i - P_i^l - \Phi^{l\,\alpha}_{i\,,\alpha} - b_i = 0,\\
&\dot{m}_{[ij]} - P^m_{[ij]} - \Phi^{m\,\alpha}_{[ij],\alpha} - \mathcal{S}^m_{[ij]} = 0,\\
&\dot{e} - P^e - \Phi^{e\alpha}{}_{,\alpha} - \mathcal{S}^e = 0,\\
&\dot{\eta} - P^\eta - J^\alpha{}_{,\alpha} - h \geqq 0,
\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}
&l_i^* = \rho^* \dot{x}_i,\\
&m^*_{[ij]} = \rho^* x_{[i} \dot{x}_{j]},\\
&e^* = \varepsilon^* + \frac{\rho^*}{2} \dot{x}_i \dot{x}_i.
\end{aligned} \tag{2.20}$$

For example, to derive $(2.19)_2$ from $(2.1)_2$, in (2.4) take

$$\psi_i^* = l_i^*, \qquad \mathcal{T}^*_{i\alpha} = T^*_{i\alpha}, \qquad \mathcal{S}_i^* = b_i^*, \qquad P_i^* = 0,$$

and define l_i, P_i^l, $\Phi^l_{i\alpha}$, b_i in a way analogous to the definitions (2.16). The balance of mass expressed by $(2.19)_1$ just states that the reference density is independent of t. I now try to write the balance of energy and the balance of moment of momentum in a form from which the inertial forces have been eliminated. To do so, I introduce the following definitions:

$$\begin{aligned}
&\bar{x}_i(Z^\alpha, 0) \equiv \frac{1}{\rho} \int_0^z \rho^+ x_i \, dZ^3,\\
&p_i(Z^\alpha, 0) \equiv x_i - \bar{x}_i,\\
&A_{ij}(Z^\alpha, 0) \equiv \varepsilon_{abc} \left[\left(\frac{\partial X^b}{\partial Z^1} \frac{\partial X^c}{\partial Z^2} T^*_{ia} p_j \right) \Big|_{Z^3=0} + \left(\frac{\partial \hat{X}^b}{\partial Z^1} \frac{\partial \hat{X}^c}{\partial Z^2} T^*_{ia} p_j \right) \Big|_{Z^3=z} \right],\\
&B_{ij\alpha} \equiv \int_0^z p_i T^+_{j\alpha} \, dZ^3,\\
&C_{ij} \equiv \int_0^z p_i b_j^+ \, dZ^3,\\
&I_{ij} \equiv \int_0^z \rho^+ p_i \dot{p}_j \, dZ^3
\end{aligned} \tag{2.21}$$

With these we can write $(2.19)_{2,3}$ as

$$\rho\ddot{\bar{x}}_i - P_i^l - \Phi^{l\,\alpha}_{i\,,\alpha} - b_i = 0,$$
$$A_{[ij]} + B^{\alpha}_{[ij],\alpha} + C_{[ij]} - \dot{I}_{[ij]} = 0.$$

It is clear from $(2.21)_1$ that $\bar{x}_i(Z_0^\alpha)$ is the mass center of the material particles situated on the material curve (2.15) in the reference configuration. Accordingly, $\rho(Z_0^\alpha)\,\dot{\bar{x}}(Z^\alpha, 0)$ equals the linear momentum associated with the material particles situated on the curve (2.15). We now separate the kinetic energy from the total energy. Set

$$\begin{aligned}
\varepsilon_m(Z^\alpha, 0) &\equiv \tfrac{1}{2}\int_0^z \rho^+\,\dot{p}_i\,\dot{p}_i\,dZ^3,\\
\varphi_\beta(Z^\alpha, 0) &\equiv \int_0^z \dot{p}_i\,T^+_{i\beta}\,dZ^3,\\
\zeta(Z^\alpha, 0) &\equiv \int_0^z \dot{p}_i\,b_i^+\,dZ^3,\\
\xi(Z^\alpha, 0) &\equiv \varepsilon_{abc}\frac{\partial X^b}{\partial Z^1}\frac{\partial X^c}{\partial Z^2}T^*_{ia}\dot{p}_i\Big|_{Z^3=0} + \varepsilon_{abc}\frac{\partial \hat{X}^b}{\partial Z^1}\frac{\partial \hat{X}^c}{\partial Z^2}T^*_{ia}\dot{p}_i\Big|_{Z^3=z},\\
q(Z^\alpha, 0) &\equiv \varepsilon_{abc}\frac{\partial X^b}{\partial Z^1}\frac{\partial X^c}{\partial Z^2}Q_a^*\Big|_{Z^3=0} + \varepsilon_{abc}\frac{\partial \hat{X}^b}{\partial Z^1}\frac{\partial \hat{X}^c}{\partial Z^2}Q_a^*\Big|_{Z^3=z}.
\end{aligned} \tag{2.23}$$

Introducing the definitions* of e, P^e, Φ^e_α and $\sigma^{\to e}$ in $(2.19)_4$, eliminating the inertial forces $\rho\ddot{\bar{x}}_i$ by substituting from $(2.22)_1$, and rewriting the resulting equation in terms of the quantities defined in (2.23), we obtain

$$\dot{\varepsilon} + \dot{\varepsilon}_m - \xi + q + \varphi^\alpha{}_{,\alpha} - Q^\alpha{}_{,\alpha} - \zeta - \gamma = 0,$$

where**

$$Q_\alpha = \int_0^z Q_\alpha^+\,dZ^3.$$

From (2.3), the defining relations (2.9), (2.16) and the definitions (2.21) and (2.23), we can deduce the transformation laws for various quantities; for example,

$$\begin{gathered}
\hat{\rho} = \rho, \qquad \hat{\bar{x}}_i = Q_{ij}\bar{x}_j + c_i, \qquad \hat{P}_i^l = Q_{ij}P_j^l,\\
\hat{p}_i = Q_{ij}p_j, \qquad \hat{A}_{ij} = Q_{ik}Q_{jl}A_{kl}, \qquad \hat{B}_{ij\alpha} = Q_{ik}Q_{jl}B_{kl\alpha},\\
\hat{I}_{ij} = Q_{ik}Q_{jl}I_{kl} + Q_{ik}E_{kl}\dot{Q}_{jl},\\
\hat{\varepsilon} = \varepsilon, \qquad \hat{\gamma} = \gamma, \qquad \hat{q} = q, \qquad \hat{Q}_\alpha = Q_\alpha,\\
\hat{\varepsilon}_m = \varepsilon_m + \dot{Q}_{ij}Q_{ik}I_{jk} + \tfrac{1}{2}\dot{Q}_{ij}\dot{Q}_{ki}E_{jk},\\
\hat{\xi} = \xi + Q_{ij}\dot{Q}_{ik}A_{jk},\\
\hat{\varphi}_\alpha = \varphi_\alpha + Q_{ij}\dot{Q}_{ik}B_{kj\alpha},
\end{gathered} \tag{2.25}$$

* These are defined in a way analogous to the definitions (2.16).
** Cf. (2.16).

where

$$E_{ij}(Z^\alpha, 0) \equiv \int_0^z \rho^+ p_i p_j \, dZ^3.$$

From $(2.21)_6$ and $(2.25)_{15}$, we get the following differential equation relating E_{ij} and I_{ij}:

$$\dot{E}_{ij} = I_{ij} + I_{ji}. \tag{2.26}$$

The preceding details make clear that the derivation of (2.22) and $(2.24)_1$ from (2.1) is rigorous. However, the information provided by these is much less than that provided by (2.1). The relation between (2.22), $(2.24)_1$ and (2.1) is somewhat similar to that between the continuum theory and the atomistic theory. As the continuum theory is designed to give information about the gross effects of deformation, similarly (2.22) and $(2.24)_1$ are expected to deliver information about suitably defined mean values of various quantities. In thin shell theory, the shell is replaced by an appropriate* inner surface and the upper and lower surfaces of the shell are parallel to it (see *e.g.*, the definition of a shell-like body in [NAGHDI, 5, Section 4]). Here C, a part of the boundary of L and V, defined by (2.7), need not equal R. However,

$$\int_C \psi \, dZ^1 \, dZ^2 = \int_V \psi^+ \, dZ^1 \, dZ^2 \, dZ^3,$$

the total ψ associated with the region V rather than with R. In those cases when either the geometry of R is such that with no choice of co-ordinate system Z, $V=R$, or the most natural choice of co-ordinates Z as dictated by the geometry of R makes $V \neq R$, we account for ψ, associated with $R-V$, by postulating an edge density ψ_e defined on ∂C such that

$$\int_C \psi \, dZ^1 \, dZ^2 + \oint_{\partial C} \psi_e \, ds = \int_R \psi^+ \, dZ^1 \, dZ^2 \, dZ^3.$$

We shall further illustrate this point in an example worked out below.

As is sometimes done for theories of shells and rods and in kinetic theory of gases, we could also have considered higher order moments of the balance laws (2.1) and deduced their averages. These higher order moments would give some information about surface couples and other related variables. For a simple body, it is not clear how one should use this extra information. We remark that for the purpose of exploring some information about the boundary conditions of surface traction and heat flux, (2.22) and $(2.24)_1$ suffice. Also, this is the usual place to stop in thin shell theory. Below we discuss some features of the balance laws (2.22), $(2.24)_1$.

In (2.19), the emergence of the production term denoted by P^l, P^e, *etc.*, which can be associated with the supply of the corresponding quantity from within L, has some interesting consequences. In COLEMAN's** thermodynamics, the fields of body force and heat supply are taken to be arbitrary, and thus the Clausius-Duhem inequality is required to hold as an identity. Also, the feasibility of eliminating the supply term from the energy equation and the entropy inequality facilitates the exploitation of the latter for obtaining restrictions on the constitutive equa-

* Usually, this is called the middle surface.

** See, *e.g.*, COLEMAN [8].

tions. In the deduced two dimensional formalism, the external source terms in the energy equation and the entropy inequality would not, in general, be related so simply, and therefore the above advantage disappears. Also, there is the possibility that $\int_0^z \gamma^+ dZ^3$ may equal zero, whereas $\int_0^z h^+ dZ^3$ does not or *vice-versa*. Thus, from the two dimensional viewpoint, L may be free from the supply of internal energy without being free from the entropy supply or *vice-versa*. Even if $h^* = 0$ and therefore $h=0$, the appearance of the term P^n in $(2.19)_5$ would mean that there is an apparent supply of entropy. This means that a loading device, which is supply free from a 3-dimensional viewpoint, would appear to be supply receiving from the 2-dimensional viewpoint. Also, even if $h^* \neq 0$, $\int_0^z h^+ dZ^3$ could be zero everywhere on C, thus implying that even if L is receiving external supply of entropy, the 2-dimensional equations would indicate that it is not. These remarks make clear that both MÜLLER's entropy principle and COLEMAN's approach to thermodynamics need to be generalised to apply to this 2-dimensional theory.

One possibility is to consider (2.22) and $(2.24)_1$ as equations of motion for the fields of position $\boldsymbol{x}$, temperature θ and possibly other fields of interest, introducing constitutive equations for the quantities $\bar{x}_i$, P_i^l, $\Phi_{i\alpha}^l$, $A_{[ij]}$, $B_{[ij]\alpha}$, $C_{[ij]}$, $I_{[ij]}$, p_i, ε, ε_m, ξ, q, φ_α, Q_α, ζ as is often done for shell, plate or membrane theories. In fact, conventional theories of thin plates or shells could provide useful approximate theories of this type. MINDLIN's* [9] work on plated crystal plates provides a simple illustration of this remark. For a fixed geometry of L, the constitutive equations for $\bar{x}$, Φ^l, *etc.*, would partially** characterize the material of L. One possibility is to derive these constitutive equations, using essentially their definitions (2.21) and (2.23), from the known 3-dimensional constitutive equations. Another is to postulate these. Usually the derivation from 3-dimensional constitutive equations presents considerable difficulties, and it is easier and simpler to follow the direct approach. This is analogous to using the theory of Cosserat surfaces to describe shells. I propose that these constitutive equations be such that the balance of moment of momentum $(2.22)_2$ is satisfied identically. Also, these constitutive relations should obey the transformation laws exemplified by (2.25) and the appropriate entropy principle. If one determined the constitutive equations from relations like (2.21), then, since quantities bearing the superscript + are densities per unit co-ordinate volume or co-ordinate area, the calculated constitutive relations would depend upon the choice of the curvilinear co-ordinate system Z. However, these quantities corresponding to different choices of the co-ordinate system Z are related to each other, the equivalence relation depending on the nature of the problem. For example, in the static problem $\ddot{\bar{x}}_i = 0$ and assuming $b_i^+ = 0$, then two $\Phi_{i\alpha}^l$'s can differ by at most a solenoidal vector field since P_i^l is determined from the surface tractions on the boundary of L. This remark suggests that we should talk about equivalence classes of constitutive equations with the equivalence relation depending on the constitutive variable and on the

* See also the remarks after equation (2.30) below.

** Because the constitutive equations also depend upon the choice of the co-ordinate system Z.

nature of the problem. As regards the smoothness of these constitutive equations, it is necessary that the flux terms be smoother than the densities. Indeed, for $(2.22)_1$ with $\ddot{\overline{x}}_i=0$ to be meaningful, P_i^l and b_i need just be defined, whereas $\Phi_{i\alpha}^l$ must be differentiable on C. The model calculations given in the next two sections illustrate the averaging procedure and the aforementioned remarks.

Because of the averaging carried out to obtain quantities like ε, $\overline{x}_i$, *etc.*, NOLL'S* definition of peer group and homogeneity is not easy to extend to the present case. If one considers from the 3-dimensional viewpoint a homogeneous deformation of a homogeneous loading device, the two dimensional constitutive equations may depend explicitly upon the surface particle virtually implying that L is in some sense inhomogeneous. The problem of material symmetry in thin elastic shells has been discussed by NAGHDI [5] and more recently by ERICKSEN [11].

To deduce information about the boundary conditions it is simplest, though not necessary, to assume that the part** F of the boundary of the loading device is isolated in the sense of the following definition:

Definition 2.1. F will be said to be *isolated* if and only if, at each point of F, there is no mechanical work done and there is no flux of energy through it.***. When F is isolated, the second term in $(2.23)_4$ and in $(2.23)_5$ vanishes and one gets a simple interpretation for ξ and q. For example, q equals the heat flux at the contact surface C. Also† P_i^l may equal the surface tractions f_i exerted by B on L at the points of the contact surface C. We recall that these tractions are measured per unit co-ordinate area. Depending upon the contact conditions at the common interface, we classify the loading device in the following two categories.

Definition 2.2. A *contact loading device* L for a body B is a 3-dimensional continuum occupying a region R in the reference configuration such that

(i) R is in the exterior of the region R_b occupied by the body B in its reference configuration,

(ii) $C\equiv\partial\overline{R}\cap\partial\overline{R}_b\neq\emptyset$,

(iii) C is a material surface for L,

(iv) F, defined by (2.6), is isolated as is made precise in Definition 2.1, and

(v) L obeys the balance laws (2.1).

In stating the next definition, we shall need the following notations. Let $g(\boldsymbol{x})$ be a field defined in two regions R_1 and R_2 of which a smooth orientable surface S is a common boundary. Granted that the limits indicated in the following rela-

* See *e.g.* TRUESDELL & NOLL [10].

** Recall definition (2.6) of F.

*** In some problems, it may suffice to assume that $F\cap V$ is isolated in the sense of the Definition 2.1.

† When F is traction free, $(2.20)_1$ and (2.13) imply

$$P_i^l(Z^\alpha,0)=\varepsilon_{abc}\frac{\partial X^b}{\partial Z^1}\frac{\partial X^c}{\partial Z^2}T_{ia}^*\bigg|_{Z^3=0}\equiv f_i. \qquad (2.27)$$

tion (2.28) exist, then

$$[g](x) \equiv \lim_{\substack{y_1 \to x \\ y_1 \in R_1}} g(y_1) - \lim_{\substack{y_2 \to x \\ y_2 \in R_2}} g(y_2), \qquad x \in S, \tag{2.28}$$

gives the jump $[g](x)$ of g at $x \in S$. The function $g(x)$ is continuous across S if and only if $[g](x)=0$ everywhere on S. Henceforth we take the reference configurations of the body and the loading device as the ones they occupy when they are just abutted together*. I also assume that both the body and the loading device are referred to the same co-ordinate system X. Thus those material particles of the body and the loading device which are situated on the common interface C will bear the same material co-ordinates X_a. Now we are ready to state the following

Definition 2.3. An *intimate-contact loading* device for B is a contact loading device which satisfies the following continuity conditions at the common interface C:

$$\begin{aligned} &[x_i(X_a, t)] = 0, \quad \forall X_a \in C, \quad t > 0, \\ &[f_i(X_a, t)] = 0, \\ &[\theta(X_a, t)] = 0, \\ &[q(X_a, t)] = 0, \end{aligned} \tag{2.29}$$

where f_i and q are the surface tractions and the heat flux per unit co-ordinate area.

In thermo-mechanical terms, $(2.29)_1$ states that there is no slip at the interface and $(2.29)_3$ implies that the contiguous parts of the body and the loading device have the same temperature. The relations $(2.29)_2$ and $(2.29)_4$ imply, respectively, the continuity of the surface tractions and the heat flux across C. With the usual assumptions on the various fields, $(2.29)_2$ and $(2.29)_4$ are equivalent to the assertion that the balance laws (2.1) hold for regions containing parts of the interface C and $(2.29)_1$ is satisfied. The condition $(2.29)_3$ certainly holds when either B or L, or both, are linearly heat conducting. The contact loading devices as defined above do not embrace all conceivable kinds of surface loading devices. For example, B could bear surface charges which can interact with a charge situated at some distance from ∂B, and thus the latter can exert surface tractions on B. Also, during the deformation process, different parts of L could come in contact with ∂B, as for example in a roller bearing or in a globule of fluid rolling over the surface of the body. Thus it is rather simple to devise examples of loading devices not covered by Definition 2.2. The following discussion is limited to exploring how the contact loading devices interact with the body.

Remembering that F is isolated**, and combining $(2.29)_2$ with $(2.22)_1$ and $(2.29)_4$ with $(2.24)_1$, we obtain

$$\begin{aligned} {}^b f_i &= -(\rho \ddot{x}_i - \Phi^{l\,\alpha}_{i\;\;,\alpha} - b_i), \\ {}^b q &= -(-\dot{\varepsilon} - \dot{\varepsilon}_m + Q^\alpha{}_{,\alpha} - \varphi^\alpha{}_{,\alpha} + \xi + \zeta + \gamma) \end{aligned} \tag{2.30}$$

* We reckon time from this instant.

** Here F is taken to be traction free.

where the upper left index b designates that the quantity is defined on the boundary particles of the body. If the constitutive quantities for the loading device are known, (2.30) gives, for part C of the boundary of B, the surface tractions and the heat flux in terms of whatever independent variables are involved. Since F is isolated, a possibility is to take the fields of displacement and temperature defined on C as the independent variables.

MINDLIN'S [9] work on plated crystal plates is easily interpretable in the present context. He discusses a purely mechanical theory and does not require that F be isolated in the sense of Definition 2.1. However, his analysis can easily be modified to incorporate the isolation condition on F. He imposes $(2.29)_1$ and $(2.29)_2$ at the common interface between the platings and the plate; thus the platings can be thought of as an intimate-contact mechanical loading device for the plate. Assuming (2.1) and linear elastic constitutive equations for the platings, he derives approximate versions of the 2-dimensional balance laws and the constitutive equations.

We now point out a peculiar feature of intimate-contact loading devices. For our discussion, the choice of independent variables is immaterial. I discuss a purely mechanical problem and take displacements defined on C as the independent variables. Assuming that the constitutive relations for the loading device are known, then, in principle, the boundary condition $(2.30)_1$ might be expected to determine the deformation of the points of the body lying in a small neighborhood of C. The deformation of these very points is also given by the field equations governing the deformation of all points of the body. That is, we have two sets of differential equations which supposedly determine the deformation of the neighborhood of C. It is quite possible that these two sets of equations are not mutually compatible. This causes a concern about the genuineness of $(2.30)_1$ as the mechanical boundary conditions for the part C of the boundary of B.

We can state the above problem in a slightly different way as follows. When the constitutive equations for the loading device are substituted in $(2.22)_1$, then in principle the field equations $(2.22)_1$ together with some side conditions are expected to determine the field of $\boldsymbol{x}$ defined on C. The known constitutive relations for L would then yield the surface tractions defined at points of C. The continuity conditions $(2.29)_{1,2}$ would make known the boundary values of surface tractions and displacements on part C of the boundary of B. This boundary data is more than can be assigned for some bodies. That should be clear from the case of the linear elastic problem treated below. To resolve this issue, we shall explore the consequences of the following proposal:

The constitutive equations of intimate-contact loading devices should be such that the balance laws (2.22), $(2.24)_1$ *are satisfied identically.* (2.31)

If we adopt the viewpoint that the fields of body force and heat supply are arbitrary, then the requirement (2.31) does not place any restrictions on the constitutive relations since we can set

$$b_i = \rho\,\ddot{\bar{x}}_i - {}^l f_i - \Phi^{l\,\alpha}_{\;i},$$

and

$$\gamma = \dot{\varepsilon} + \dot{\varepsilon}_m \quad {}^l q + \varphi^{\alpha}{}_{,\alpha} - Q^{\alpha}{}_{,\alpha} - \xi - \zeta.$$

However, when b_i and γ are not arbitrary, *e.g.*, when the loading device is supply free, *i.e.*, $b_i=0$, $\gamma=0$, then the constitutive equations for $\bar{x}_i$, $\Phi_i{}^l{}_\alpha$ *etc.* are severely restricted by the requirement (2.31). There is nothing in our experiential background which can provide a motivation for (2.31). For some simple examples of loading devices, we can derive, by mechanistic calculations, the surface constitutive equations from their known three dimensional counterparts. For these cases, we can check the validity of (2.31).

3. Model Calculations: Thermal Problem

We now illustrate the averaging method and illuminate some of the aforementioned remarks by working out the details for the case when the intimate contact loading device is a supply free, homogeneous, isotropic, linear thermoelastic half space D. Since rather few results are known for coupled linear thermoelasticity, I shall consider the "uncoupled" case. That is, the thermal and mechanical problems are considered separately. Further the deformations envisaged are "quasi-static" in the sense that unsteady thermal problems are discussed but the mechanical problem studied is a static one. Unless that part of the boundary of the adjoining body B which is glued to D equals ∂D, the common interface C between the body and the loading device would be a proper subset of the bounding plane ∂D of D. Take reference configurations for B and D as those occupied by them at the instant of glueing, and reckon time from this instant. We assume the reference configuration for D to be a natural one, *i.e.* the initial stress vanishes, and the initial temperature = constant $\theta_0>0$, but make no assumptions regarding either the constitutive equation for B or its reference configuration. However, for consistency with the assumption that D is an intimate contact loading device, surface tractions vanish and $\theta(X,0)=\theta_0$ at those points of B which lie on C. Here, it is convenient to choose a rectangular Cartesian coordinate system for D such that the X_3-axis projects into D and $X_3=0$ on the bounding plane ∂D. In the notations of the previous section, the X-axes coincide with the Z-coordinate curves and, to avoid confusion between superscript and power, we shall use subscripts only for labelling the coordinates. Thus a typical material tube T would be a material cylinder with the cross-sections contained in the planes X_3 = constant and generators parallel to the X_3-axis.

I now consider a purely thermal problem. Let B and D exchange heat through the contact area C and we investigate this transfer of heat when the temperature deviations

$$v(X,t)\equiv\frac{1}{\theta_0}(\theta(X,t)-\theta_0), \qquad X\in D,$$

are small. The only relevant balance laws are the energy equation and the entropy inequality. The former, when linearized around $\theta=\theta_0$, takes the form $(3.1)_1$ in terms of the temperature deviation v:

$$c\frac{dv}{dt}(X,t)=\kappa\nabla^2 v(X,t),$$

where

$$c \equiv \frac{d\varepsilon}{d\theta}\bigg|_{\theta=\theta_0}$$
$$\equiv -\frac{1}{3}\frac{\partial Q_a}{\partial \theta, a}\bigg|_{\theta=\theta_0,\,\theta,a=0}$$

Here c and κ are absolute constants since D is presumed to be homogeneous and isotropic. c is called the specific heat and κ the thermal conductivity at temperature θ_0. It is common to assume, and I also do, that c and κ are positive. On C, $(2.29)_3$ and $(2.29)_4$ hold and Definition 2.1 for the isolation condition on F takes the form $(3.2)_{1,2}$. Thus the side conditions governing the solution of $(3.1)_1$ are

$$\begin{aligned}
&q(X,t)\to 0 && \text{as } |X|\equiv(X_i X_i)^{\frac{1}{2}}\to\infty, \quad t>0,\\
&q(X,t)=0 && (X,t)\in(\partial D-C)\times(0,t],\\
&q(X,t)={}^l q(X,t) && (X,t)\in C\times(0,t],\\
&v(X,0)=0 && X\in D.
\end{aligned}\tag{3.2}$$

In $(3.2)_3$ and below, the upper left index l prefixed to a quantity signifies the value of the corresponding quantity for the loading device; q is the heat flux* per unit Euclidean surface area. Introducing the notations

$$r^2\equiv(X_\alpha-X'_\alpha)(X_\alpha-X'_\alpha)+X_3^2,$$
$$H_n(X,X',t,\tau)\equiv[\exp(-c\,r^2/4\kappa(t-\tau))]/(t-\tau)^{\frac{n}{2}} \qquad (n=1,2,$$

we find the solution** of $(3.1)_1$ satisfying (3.2) to be

$$v(X,t)=-\frac{1}{4}\left(\frac{c}{\pi\kappa}\right)^{\frac{3}{2}}\int_0^t d\tau\int_C H_3(X,X',t,\tau)\,{}^l q(X',\tau)\,dA(X'),$$

where

$$dA(X')\equiv dX'_1\,dX'_2$$

is the surface element of area on C. The integral in (3.4) is improper, the integrand having a singularity at $X'_\alpha=X_\alpha$, $X_3=0$, $\tau=t$. However, the singularity is integrable. Indeed, writing $H_n(X,X',t,\tau)$ in the form

$$\begin{aligned}H_n(X,X',t,\tau)=&(t-\tau)^{-p}(r^2)^{p-\frac{n}{2}}[r^2/(t-\tau)]^{\frac{n}{2}-p}\\
&\times\exp.(-c\,r^2/4\kappa(t-\tau)),\qquad 0<p<1,\end{aligned}$$

we get

$$|H_n(X,X',t,\tau)|\leq\frac{\text{Constant}}{(t-\tau)^{p_r n-2p}},$$

and the right hand side is integrable. Henceforth, we shall denote the points of $D-\partial D$ by X, and a point on ∂D will be designated by $\bar{X}$. Thus, for the surface

* *Cf.* $(2.23)_5$ and the present choice of the co-ordinate system.

** See *e.g.* CARSLAW & JAEGER [2].

temperature, (3.4) gives

$$
\begin{aligned}
{}^{l}v(\overline{X}, t) &\equiv \lim_{X\to\overline{X}} v(X, t) \\
&= \frac{1}{4}\left(\frac{c}{\pi\kappa}\right)^{\frac{3}{2}} \int_0^t d\tau \int_C H_3(\overline{X}, X', t, \tau)\, {}^{l}q(X', \tau)\, dA(X'),
\end{aligned}
$$

where

$$H_n(\overline{X}, X', t, \tau) = [\exp.(-c\, r_0^2/4\kappa(t-\tau))]/(t-\tau)^{\frac{n}{2}}, \qquad n=1,2,$$

and

$$r_0^2(\overline{X}, X') \equiv (\overline{X}_\alpha - X'_\alpha)(\overline{X}_\alpha - X'_\alpha).$$

Using Fourier's law,

$$Q^* = -\kappa\,\theta_0\, \mathrm{Grad}\, v,$$

and observing that, for the present choice of co-ordinates X,

$$g_{ij} = \delta_{ij}$$

and

$$z(\overline{X}) = \infty \qquad \forall\, \overline{X} \in \partial D,$$

we see from $(2.24)_2$ that the flux of energy is given by

$$Q_\beta = \int_0^\infty Q_\beta^*\, dX_3 = -\kappa\,\theta_0 \int_0^\infty \frac{\partial V}{\partial X_\beta}\, dX_3.$$

Upon substitution from (3.4) and carrying out the integration*, we obtain

$$Q_\beta(\overline{X}, t) = -\frac{c^2\theta_0}{8\kappa\pi} \int_0^t d\tau \int_C (\overline{X}_\beta - X'_\beta)\, H_4(\overline{X}, X', t, \tau)\, {}^{l}q(X', \tau)\, dA(X'). \tag{3.8}$$

When the internal energy density is normalised by setting $\varepsilon^*(\theta_0)=0$, its value for the linear heat conductor is given by

$$\varepsilon^*(X, t) = c\,\theta_0\, v(X, t),$$

and this, when combined with (3.4) and an expression of the type $(2.16)_2$, gives the following expression for the surface density of internal energy:

$$\varepsilon(\overline{X}, t) = -\frac{c^2\theta_0}{4\pi\kappa} \int_0^t d\tau \int_C H_2(\overline{X}, X', t, \tau)\, {}^{l}q(X', \tau)\, dA(X'). \tag{3.9}$$

To demonstrate the existence of the derivatives of Q_β and ε, we observe that (3.8), when differentiated with respect to $\overline{X}_\alpha$, and (3.9), when differentiated with respect to time t, yield

$$
\begin{aligned}
Q_{\beta,\alpha}(\overline{X}, t) &= -\frac{c^2\theta_0}{8\kappa\pi} \int_0^t d\tau \int_C (\overline{X}_\beta - X'_\beta)\, H_4(\overline{X}, X', t, \tau)\, \frac{\partial\, {}^{l}q}{\partial X_\alpha}(X', \tau)\, dA(X'), \\
\dot{\varepsilon}(\overline{X}, t) &= -\frac{c^2\theta_0}{4\pi\kappa} \int_0^t d\tau \int_C H_2(\overline{X}, X', t, \tau)\, {}^{l}\dot{q}(X', \tau)\, dA(X').
\end{aligned} \tag{3.10}
$$

* Here we use the known result $\int_0^\infty \exp.(-y)\, \frac{1}{y^{\frac{1}{2}}}\, dy = \sqrt{\pi}$.

To obtain these, integration by parts has been performed which is facilitated by the fact that the kernel is of the convolution type. Also ${}^l q$ is assumed to be differentiable on C and continuous across ∂C. If the latter condition does not hold, a term involving the line integral along ∂C would appear in (3.10). The relations (3.8) and (3.9) establish linear representations for Q_β and ε, and (3.10) depicts the chain rule of differentiation of Q_β and ε. To calculate η and J_β, it is sufficient to point out that, with the normalization $\eta^*(\theta_0)=0$, we have

$$\eta(\overline{X}, t)=\frac{1}{\theta_0}\,\varepsilon(\overline{X}, t)$$

and

$$J_\beta(\overline{X}, t)=\frac{1}{\theta_0}\,Q_\beta(\overline{X}, t).$$

Thus representations similar to (3.8), (3.9), $(3.10)_1$ and $(3.10)_2$ will result for J_β, η, $J_{\beta,\alpha}$ and $\dot{\eta}$, respectively. It is worth mentioning that the above linear representations for ε, Q_β, *etc.* are not unique since one could integrate these by parts and obtain linear representations in terms of first or higher derivatives of ${}^l q$. Depending upon the smoothness properties of ${}^l q(X, t)$, this process would introduce line integrals along ∂C. For example when ∂C has a representation

$$\overline{X}_\alpha=\overline{X}_\alpha(s),$$

s being its arc length, (3.8) can equivalently be written as

$$Q_\beta(\overline{X}, t)=-\frac{c}{4\pi}\left[\int_0^t d\tau \int_C H_2(\overline{X}, X', t, \tau)\frac{\partial^l q}{\partial X'_\beta}(X', \tau)\, dA(X')\right.$$
$$\left.-\varepsilon_{\beta\alpha}\int_0^t d\tau \oint_{\partial C} H_2(\overline{X}, X', t, \tau)\,{}^l q(X', \tau)\frac{dX'_\alpha}{ds}\, ds\right]. \tag{3.11}$$

When ${}^l q$ is continuous across ∂C, the second integral vanishes; otherwise it gives a contribution. It may be remarked that the kernel in (3.11) is somewhat smoother than that in (3.8).

It is a simple matter to verify that these constitutive relations are invariant with respect to the translation of the time axis. This just illustrates the freedom of choosing the origin of the time scale and is a requirement for the constitutive equations to be frame-indifferent.

With the above choice of the co-ordinate system, $\int_C \varepsilon\, dA$ would equal the internal energy of the solid cylinder

$$V\equiv\{X\colon X\in D,\ (X_\alpha, 0)\in C\}.$$

At the risk of repetition, I remark that the energy of the remaining portion of D is accounted for by postulating an edge density ε_e defined on ∂C. To demonstrate its representation, it is convenient to introduce on ∂D a curvilinear coordinate system (Y_1, Y_2) defined by

$$\overline{X}_\alpha=\overline{X}_\alpha(Y_\beta),$$

such that

$$\det\left|\frac{\partial\overline{X}_\alpha}{\partial Y_\beta}\right|\neq 0,$$

everywhere on ∂D and ∂C is given by

$$Y_1 = \text{constant, say} = 1.$$

Thus at points of ∂C we can define the edge density by the relation

$$\varepsilon_e(1, Y_2, t) \equiv \int^{\infty} \varepsilon(Y_1, Y_2)\, dY_1,$$

$$\frac{c^2\theta_0}{4\pi\kappa} \int_1^{\infty} dY_1 \int_0^t d\tau \int_C \frac{e^{-\frac{c\, r_0^2(Y_\beta, Y'_\beta)}{4\kappa(t-\tau)}}}{(t-\tau)}\, {}^l q(Y'_\beta, \tau)\, dA(Y'_\beta),$$

where

$$dA(Y'_\beta) = \frac{\partial \overline{X}_1}{\partial Y_\delta}(Y'_\beta) \frac{\partial \overline{X}_2}{\partial Y_\alpha}(Y'_\beta)\, dY'_\delta\, dY'_\alpha,$$

$$r_0(Y_\beta, Y'_\beta) = |X_\alpha(Y_\beta) - X_\alpha(Y'_\beta)|.$$

Hence ε and ε_e would satisfy

$$\int_C \varepsilon\, dA + \oint_{\partial C} \varepsilon_e\, dY_2 = \int_D \varepsilon^*\, dV.$$

The preceding details also make clear that, had we chosen the material tubes differently, the expression for ${}^l v(\overline{X}_\alpha, t)$ alone would tally with (3.6); all others in general would have a different form. If we denote by $\tilde{\varepsilon}$ the difference in ε evaluated by two different choices of material tubes and use like notation for Q_β, since these satisfy the appropriate form of $(2.24)_1$, the equivalence relation would be

$$\dot{\tilde{\varepsilon}} - \tilde{Q}^\beta{}_{,\beta} = 0.$$

In order to derive the boundary conditions for the body from $(2.29)_{3,4}$ and (3.6), I assume that part C of the boundary of B is described by previously selected rectangular Cartesian co-ordinates. Thus an outer unit normal to ∂B at points of C will have components (0, 0, 1). From $(2.29)_{3,4}$ and (3.6), we get

$$ {}^b v(\overline{X}, t) = \frac{1}{4}\left(\frac{c}{\pi\kappa}\right)^{\frac{1}{2}} \int_0^t d\tau \int_C H_3(\overline{X}, X', t, \tau)\, {}^b q(X', \tau)\, dA(X'), \tag{3.13}$$

which expresses a thermal boundary condition for part C of the boundary of B. This is not of the classical type; it states that the temperature at a point $\overline{X}$ of C is affected by the field of heat flux over C and not just in the neighborhood of $\overline{X}$.

We note that the constants c and κ in (3.13) are the values of the specific heat and the thermal conductivity for the loading device. Some of the thermal boundary conditions stated by CARSLAW & JAEGER [2, Chapter 1] are of the form

heat flux = function of surface temperatures.

We can obtain this type of functional dependence by inverting (3.13). However, the inversion of (3.13) does not appear to be straightforward. Accordingly, I seek a solution of $(3.1)_1$ under the boundary conditions

$$\begin{aligned} v(X, t) &= 0, && (X, t) \in (\partial D - C) \times (0, t], \\ &= {}^l v(X, t) && (X, t) \in C \times (0, t], \\ &\to 0 && \text{as } |X| \to \infty, \quad t > 0, \end{aligned} \tag{3.14}$$

and the initial condition $(3.2)_4$. The solution* is

$$v(X,t)=\frac{X_3}{8}\left(\frac{c}{\pi\kappa}\right)^{\frac{3}{2}}\int_0^t d\tau\int_C H_5(X,X',t,\tau)\,{}^l v(X',\tau)\,dA(X'). \tag{3.15}$$

Since the boundary conditions on $\partial D-C$ are different in the two cases, (3.15) is not the inverse of (3.4). However, it is clear from (3.15) and (3.7) that the heat flux vanishes at ∞, so that the derivation of the boundary condition $(2.30)_2$ from the balance law $(2.24)_1$ holds. Comparing (3.15) with (3.4), we see that the kernel in (3.15) has a higher order singularity. This suggests that the representations for ε, $\boldsymbol{Q}$, *etc.* in terms of temperature distribution on C would be less smooth than that in terms of heat flux. It seems worth mentioning that, in (3.4), the order of integration is immaterial since the singularity of the integrand is absolutely integrable in each variable separately**. However, the integral in (3.15) is a repeated integral, and thus only the τ-integral is taken as an improper integral. We now proceed to calculate heat flux at points of C. From (3.7), the definitions of q and $v_{,3}$, and the fact that outer unit normal to ∂D has components $(0,0,-1)$, we have

$${}^l q(X_\alpha,0,t)=\kappa\theta_0\lim_{X_3\to 0^+}\frac{v(X_\alpha,X_3,t)-v(X_\alpha,t)}{X_3} \tag{3.16}$$

Assuming that v is differentiable almost everywhere on ∂D, we introduce the notations

$$\begin{aligned}
{}^l\check{v}(X_\alpha,X'_\alpha,t,\tau)&\equiv{}^l v(X'_\alpha,\tau)-{}^l v(X_\alpha,t),\\
{}^l\check{\check{v}}(X_\alpha,X'_\alpha,t,\tau)&\equiv{}^l v(X'_\alpha,\tau)-{}^l v(X_\alpha,t)-(X_\beta-X'_\beta)\,\frac{\partial\,{}^l v(X_\alpha,t)}{\partial X_\beta}.
\end{aligned} \tag{3.17}$$

Before evaluating the limit in (3.16), we note that, because of $(3.14)_1$, we can replace the region C of integration in (3.15) by ∂D. Doing so and substituting for v in (3.16), we obtain

$$\begin{aligned}
{}^l q(X_\alpha,0,t)=\kappa\theta_0\lim_{X_3\to 0^+}&\left[\frac{{}^l v(X_\alpha,t)}{X_3}\left\{\frac{X_3}{8}\left(\frac{c}{\pi\kappa}\right)^{\frac{3}{2}}\int_0^t d\tau\int_{\partial D}H_5(X,X',t,\tau)\,dA(X')-1\right\}\right.\\
&+\frac{1}{8}\left(\frac{}{\pi\kappa}\right)^{\frac{3}{2}}\int_0^t d\tau\int_{\partial D}H_5(X,X',t,\tau)\,{}^l\check{\check{v}}(X_\alpha,X'_\alpha,t,\tau)\,dA(X'\\
&\left.+\frac{}{8}\left(\frac{c}{\pi\kappa}\right)^{\frac{3}{2}}\frac{\partial\,{}^l v(X_\alpha,t)}{\partial X_\beta}\int_0^t d\tau\int_{\partial D}H_5(X,X',t,\tau)(X_\beta-X'_\beta)\,dA(X')\right]
\end{aligned} \tag{3.18}$$

A simple calculation shows that

$$\begin{gathered}
\lim_{X_3\to 0}\left[\frac{1}{X_3}\left\{\frac{X_3}{8}\left(\frac{c}{\pi\kappa}\right)^{\frac{3}{2}}\int_0^t d\tau\int_{\partial D}H_5(X,X',t,\tau)\,dA(X'_\alpha\qquad\right\}\right]\qquad\left(\frac{c}{\kappa t}\right)\\
\int_0^t d\tau\int_{\partial D}H_5(X,X',t,\tau)(X_\alpha-X'_\alpha)\,dA(X')=0.
\end{gathered} \tag{3.19}$$

* See Carslaw & Jaeger [2].

** *Cf.* (3.5).

Thus for every $t>0$, the limit of the first term on the right hand side of (3.18) exists and the third term in (3.18) vanishes. Since

$$\int_0^t d\tau \int_{\partial D} (\ )\, dA(X') = \int_0^t d\tau \int_C (\ )\, dA(X') + \int_0^t d\tau \int_{\partial D - C} (\ )\, dA(X'),$$

and, at points of $\partial D - C$,

$${}^{l}\check{v}(X_\alpha, X'_\alpha, t, \tau) = -{}^{l}v(X_\alpha, t) - (X_\beta - X'_\beta)\, \frac{\partial\, {}^{l}v(X_\alpha, t)}{\partial X_\beta}.$$

we can rewrite (3.18) as

$$\begin{aligned} {}^{l}q(\bar{X}_\alpha, t) = & -H(\bar{X}, t)\,{}^{l}v(\bar{X}, t) - K_\beta(\bar{X}, t)\, \frac{\partial\, {}^{l}v(X_\alpha, t)}{\partial X_\beta}\bigg|_{X_\alpha = \bar{X}_\alpha} \\ & + \frac{1}{8}\left(\frac{c}{\pi\kappa}\right)^{\frac{3}{2}} \theta_0 \kappa \int_0^t d\tau \int_C H_5(\bar{X}, X', t, \tau)\,{}^{l}\check{v}(\bar{X}, X', t, \tau)\, dA(X'). \end{aligned}$$

Here

$$H(\bar{X}, t) \equiv +\kappa \left(\frac{c}{\kappa t}\right)^{\frac{1}{2}} \theta_0 + \frac{1}{8}\left(\frac{c}{\pi\kappa}\right)^{\frac{3}{2}} \theta_0 \kappa \int_0^t d\tau \int_{\partial D - C} H_5(\bar{X}, X', t, \tau)\, dA(X'),$$

$$K_\alpha(\bar{X}, t) \equiv +\frac{1}{8}\left(\frac{c}{\pi\kappa}\right)^{\frac{3}{2}} \theta_0 \kappa \int_0^t d\tau \int_{\partial D - C} (\bar{X}_\alpha - X'_\alpha)\, H_5(\bar{X}, X', t, \tau)\, dA(X'.$$

It may be remarked that, in (3.21), the region $\partial D - C$ of integration is unbounded. Since the integrand tends to zero exponentially as $r_0 \to \infty$, the integrals in (3.21) converge and therefore H and K_α are well defined. Combining $(2.29)_{3,4}$ with (3.20), we get

$$\begin{aligned} {}^{b}q(\bar{X}, t) = & H(\bar{X}, t)\,{}^{b}v(\bar{X}, t) + K_\alpha(\bar{X}, t)\, \frac{\partial\, {}^{b}v(X, t)}{\partial X_\alpha}\bigg|_{X = \bar{X}} \\ & - \frac{1}{8}\left(\frac{c}{\pi\kappa}\right)^{\frac{3}{2}} \theta_0 \kappa \int_0^t d\tau \int_C H_5(\bar{X}, X', t, \tau)\,{}^{b}\check{v}(\bar{X}, X', t, \tau)\, dA(X'), \end{aligned} \tag{3.22}$$

as the thermal boundary condition for part C of the boundary of B. Recalling that $v(X, t)$ is a measure of the deviation of the present temperature from the temperature in the reference configuration, (3.22) makes clear that, when we account for the heat conduction in the surroundings of B, the heat flux at the boundary of B is not given by a relation analogous to that obtained when Newton's law of cooling is assumed.

If we should calculate the linear representations for Q_α using (3.15) in place of (3.4), we should find the singularity of its kernel to be of lower order than that of the kernel in the representation (3.20) of ${}^{l}q$. This would prove the conjecture that flux terms are smoother than the densities for the linear heat conductor.

4. Model Calculations: Mechanical Problem

For the loading device considered in the preceding section, we now discuss a purely mechanical static problem. That is, we assume that the deformations of the half space D are time-independent and either the entropy or the temperature remains constant in space. Envisage infinitesimal deformations of D from its natural configuration to another configuration of equilibrium. Since the loading device is assumed to be supply free and part F of its boundary isolated in the sense of Definition 2.1, D can be deformed either by applying surface tractions on the part of the boundary of B not glued to D or by varying the body force field in B. These deformations of D are governed by

$$\operatorname{Grad}\operatorname{Div}\boldsymbol{u}+\frac{2\mu}{\lambda+\mu}\Delta\boldsymbol{u}=\boldsymbol{0}, \tag{4.1}$$

where

$$\boldsymbol{u}(X)\equiv\boldsymbol{x}(X)-X$$

is the displacement of point X, Δ is the Laplacian operator and λ, μ are Lamé's constants. Recalling Definition 2.3 of an intimate contact loading device, the following boundary conditions for D ensure that F is isolated in the sense of Definition 2.1:

$$\begin{aligned} \boldsymbol{f}(X)&=\boldsymbol{0} && X\in\partial D-C,\\ \boldsymbol{f}(X)&={}^{l}\!\boldsymbol{f}(X) && X\in C,\\ |\boldsymbol{f}(X)|&=o(1) && |X|\to\infty. \end{aligned} \tag{4.2}$$

I assume λ and μ to be such that $\mu\neq 0$ and Poisson's ratio* $\sigma\neq 1$. Then (4.1) is uniformly elliptic. If we assume that λ, μ and $(4.2)_{1,2}$ satisfy the complementing condition** on ∂D and

$$\begin{gathered} \int_C |{}^{l}\!\boldsymbol{f}(X')|\,dA(X'), \qquad \int_C |X'\otimes{}^{l}\!\boldsymbol{f}(X')|\,dA(X'),\\ \int_C |X'\otimes X'\otimes{}^{l}\!\boldsymbol{f}(X')|\,dA(X') \quad\text{and}\quad \int_C |X'\otimes X'\otimes X'\otimes{}^{l}\!\boldsymbol{f}(X')|\,dA(X') \end{gathered} \quad {}^{***} \tag{4.3}$$

are finite, then the theorem of THOMPSON [12, Thm. 7] makes $(4.2)_3$ redundant. Rather, it delivers a stronger result that $|\boldsymbol{f}(X)|$ is $o(|X|^{-3})$ as $|X|\to\infty$. Usually, in a traction boundary value problem, the surface tractions are prescribed *a priori*, and one can possibly evaluate the integrals in (4.3). But, for the present problem, this is not feasible since here the surface tractions are variables of interest. However, I choose to assume $(4.2)_3$ and thus not utilize this sophisticated result of THOMPSON. Then the theorem given by KNOPS & PAYNE [13, Thm. 6.2.2] guarantees that the solution of (4.1) under the boundary conditions (4.2) and with $\mu\neq 0$, $\sigma\neq 1$ is unique to within an infinitesimal rigid body deformation. The following explicit form of the solution is taken from SOLOMON† [14, p. 610]:

$$\begin{aligned} u_\alpha=\frac{1}{4\pi\mu}\int_C\Bigg[&\frac{\delta_{i\alpha}}{r}+\left(\frac{\delta_{i\beta}(X_\beta-X'_\beta)+\delta_{i3}X_3}{r^3}\right)(X_\alpha-X'_\alpha)\\ &+\frac{\mu}{\lambda+\mu}\frac{\partial}{\partial X_i}\left(\frac{X_\alpha-X'_\alpha}{X_3+r}\right)\Bigg]{}^{l}\!f_i\,dA(X'),\\ u_3=\frac{1}{4\pi\mu}\int\Bigg[&\frac{\delta_{i3}}{r}+\left(\frac{\delta_{i\beta}(X_\beta-X'_\beta)+\delta_{i3}X_3}{r^3}\right)X_3\\ &+\frac{\mu}{\lambda+\mu}\frac{\partial}{\partial X_i}\log(r+X_3)\Bigg]{}^{l}\!f_i\,dA(X'), \end{aligned}$$

* $\sigma=\lambda/2(\lambda+\mu)$.

** See THOMPSON [12] for the definition.

*** The symbol $\otimes$ denotes tensorial multiplication of two vectors.

† See also LOVE [15, p. 242].

where r and dA are defined as in Section 3. It is clear from (4.4) that the kernel has a singularity of the type $1/r$, which is integrable. Thus, for the surface displacements, using

$$\boldsymbol{u}(\bar{X}) \equiv \lim_{X \to \bar{X}} \boldsymbol{u}(X),$$

we obtain

$$'u_\alpha(\bar{X}) = \frac{1}{4\pi\mu} \int_C \left[\frac{\lambda+2\mu}{\lambda+\mu} \frac{\delta_{i\alpha}}{r_0} - \delta_{i3} \frac{\mu}{\lambda+\mu} \frac{(\bar{X}_\alpha - X'_\alpha)}{r_0^2} + \delta_{i\beta} \frac{\lambda}{\lambda+\mu} \frac{(\bar{X}_\alpha - X'_\alpha)(\bar{X}_\beta - X'_\beta)}{r_0^3} \right] {}'f_i(X')\, dA(X'),$$

$$'u_3(\bar{X}) = \frac{1}{4\pi\mu} \int_C \left[\frac{(\lambda+2\mu)}{(\lambda+\mu)} \frac{\delta_{i3}}{r_0} + \delta_{i\alpha} \frac{\mu}{(\lambda+\mu)} \frac{(\bar{X}_\alpha - X'_\alpha)}{r_0^2} \right] {}'f_i(X')\, dA(X').$$

Before calculating the flux of linear momentum from

$$\Phi^l_{i\alpha} = \int_0^\infty T_{i\alpha}\, dX_3, \tag{4.6}$$

we remark that, because of the infinitesimal deformations envisaged and the assumption that the reference configuration is a natural state, the Cauchy stress equals the Kirchhoff stress. Using

$$T_{ij} = \lambda u_{k,k} \delta_{ij} + 2\mu u_{(i,j)} \qquad * (4.7)$$

and (4.4), we can calculate the stress at any interior point of D. When these are substituted in (4.6) and the line integral is evaluated, we obtain the expression for flux $\Phi^l_{i\alpha}$ of linear momentum. The calculations are elementary but lengthy and yield the following for the flux $\Phi^l_{3\alpha}$:

$$\Phi^l_{3\alpha}(\bar{X}) = -\frac{1}{2\pi} \int_C \left[\frac{(\bar{X}_\alpha - X'_\alpha)}{r_0^2} \delta_{i3} + \frac{(\bar{X}_\beta - X'_\beta)(\bar{X}_\alpha - X'_\alpha)}{r_0^3} \delta_{i\beta} \right] {}'f_i(X')\, dA(X'), \tag{4.8}$$

and similar expressions would result for other components of Φ^l. Differentiation of (4.8) yields

$$\Phi^l_{3\alpha,\gamma}(\bar{X}) = -\frac{1}{2\pi} \int_C \left\{ \frac{(\bar{X}_\alpha - X'_\alpha)}{r_0^2} \delta_{i3} + \frac{(\bar{X}_\beta - X'_\beta)(\bar{X}_\alpha - X'_\alpha)}{r_0^3} \delta_{i\beta} \right\} \frac{\partial\, {}'f_i}{\partial X_\gamma}(X')\, dA(X'), \tag{4.9}$$

where we have integrated by parts and assumed that $'f$ is differentiable on C and continuous across ∂C. If the latter is not the case, a line integral representing the contribution of the discontinuity of forces across ∂C would appear in (4.9). Since

$$\varepsilon^* = \tfrac{1}{2} \lambda (u_{k,k})^2 + 4\mu u_{(i,j)} u_{(i,j)},$$

the expressions of the type $(2.16)_2$ for ε and $(3.11)_1$ for ε_e are not simple. When λ and μ satisfy

$$3\lambda + 2\mu \geqq 0, \qquad \mu \geqq 0,$$

---

* Indices in round brackets indicate that a tensor is symmetrized with respect to these indices.

ε^* would be positive definite, and therefore we can conclude that ε and ε_e would also be positive definite. For a different choice of material tubes, (4.5) would be unaltered, whereas, in general, different expressions would result for $\Phi^l_{i\alpha}$, and ε_e. For the present problem $\ddot{x}_i=0$ in $(2.22)_1$, and accordingly the equivalence relation for $\Phi^l_{i\alpha}$ would be

$$\overset{\mathrm{I}}{\Phi}{}^l_{i\alpha}\simeq\overset{\mathrm{II}}{\Phi}{}^l_{i\alpha}\Leftrightarrow\big(\overset{\mathrm{I}}{\Phi}{}^l_{i\alpha}-\overset{\mathrm{II}}{\Phi}{}^l_{i\alpha}\big)_{,\alpha}=0.$$

Since D is assumed to be permanently glued to B, $(2.29)_1$ and $(2.29)_2$ hold at the contact surface C, and, consequently, from (4.5) one can deduce the boundary conditions for part C of the boundary of B. These express surface displacements in terms of surface tractions. In [1, 3, 4] the surface tractions are stated as a function of the surface displacements. In order to obtain such a functional relationship in the present context, it would be desirable to invert (4.5). However, the inversion does not appear to be straightforward and therefore I consider the solution of (4.1) under the boundary conditions

$$\begin{aligned} \boldsymbol{u}(X)&=0 && X\in\partial D-C,\\ \boldsymbol{u}(X)&={}^l\boldsymbol{u}(X), && X\in C,\\ |\boldsymbol{u}(X)|&=o(1) && |X|\to\infty. \end{aligned}\tag{4.10}$$

When λ and μ are restricted to be such that

$$\mu\neq0,\quad \sigma\neq\ \tfrac{1}{2},\ \tfrac{3}{4},$$

the theorem given by Knops & Payne [13, Thm. 6.2.1] implies that the solution of (4.1), (4.10) is unique. The explicit form of the solution is [15, page 240]

$$u_i(X)=\frac{1}{2\pi}\int_C\left[\frac{\partial\frac{1}{r}}{\partial X_3}\delta_{ik}+\nu X_3\frac{\partial^2\left(\frac{1}{r}\right)}{\partial X_i\,\partial X_k}\right]{}^l u_k(X')\,dA(X'),\tag{4.11}$$

where

$$\nu\equiv\frac{\lambda+\mu}{\lambda+3\mu}$$

Here the kernel has a singularity of the type $1/r^2$. To calculate surface tractions, we shall need the values of displacement gradients at points of C. Were we to differentiate under the integral sign in (4.11), the singularity of the kernel would be of the type $1/r^3$, which is not integrable. However, this does show that stresses and therefore surface tractions would vanish at ∞. Recalling the derivation of the boundary condition $(2.30)_1$ from $(2.22)_1$, we see that, for the problem defined by (4.1) and (4.10), $(2.30)_1$ still holds.

At points of C, $u_{i,\alpha}$ can be calculated from the boundary data $(4.10)_2$. We now proceed to evaluate $u_{i,3}$ at points of C. We remark that because of $(4.10)_1$, the region C of integration in (4.11) can be replaced by ∂D. Also, a simple but lengthy calculation shows that

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{\partial D} -\frac{\partial \frac{1}{r}}{\partial X_3}\, dA(X') = 1\\
&\int_{\partial D} (X_\alpha - X'_\alpha)\frac{\partial \frac{1}{r}}{\partial X_3}\, dA(X') = 0,\\
&\int_{\partial D} \frac{\partial^2 \frac{1}{r}}{\partial X_i \partial X_k}\, dA(X') = 0,\\
&\frac{}{2\pi}\int_{\partial D} (X_\alpha - X'_\alpha)\frac{\partial^2 \left(\frac{1}{r}\right)}{\partial X_i \partial X_k}\, dA(X') = 2\delta_{3\,(i}\delta_{k)\,\alpha}.
\end{aligned}
\tag{4.12}
$$

By definition,

$$
{}^{l}u_{i,3}(X_\alpha, 0) = \lim_{X_3 \to 0^+} \frac{u_i(X_\alpha, X_3) - u_i(X_\alpha, 0)}{X_3}
$$

Substituting for u_i from (4.11), we get

$$
{}^{l}u_{i,3}(X_\alpha, 0) = \lim_{X_3 \to 0^+}
\left[\frac{1}{X_3}\left\{\frac{1}{2\pi}\int_{\partial D}\left(-\frac{\partial \frac{1}{r}}{\partial X_3}\delta_{ik} + \nu X_3 \frac{\partial^2\left(\frac{1}{r}\right)}{\partial X_i \partial X_k}\right){}^{l}u_k(X')\, dA(X') - u_i(X_\alpha, 0)\right\}\right].
\tag{4.13}
$$

Assuming that ${}^{l}u_i$ is differentiable almost everywhere on ∂D, we introduce the definition

$$
{}^{l}\check{\tilde{u}}_i(X_\alpha, X'_\alpha) = {}^{l}u_i(X'_\alpha, 0) - {}^{l}u_i(X_\alpha, 0) - (X_\beta - X'_\beta)\frac{\partial\, {}^{l}u_i(X_\alpha, 0)}{\partial X_\beta}.
\tag{4.14}
$$

Now making use of the relations (4.12), we can rewrite (4.13) as

$$
\begin{aligned}
{}^{l}u_{i,3}(X_\alpha, 0) &= \lim_{X_3 \to 0^+}\left[\frac{1}{X_3}\left\{\frac{1}{2\pi}\int_{\partial D}\left[-\frac{\partial \frac{1}{r}}{\partial X_3}\delta_{ik} + \nu X_3\frac{\partial^2\left(\frac{1}{r}\right)}{\partial X_i \partial X_k}\right]\right.\right.\\
&\qquad \left.\left.\times\, {}^{l}\check{\tilde{u}}_k(X_\alpha, X'_\alpha)\, dA(X'_\alpha) + 2\nu X_3 \frac{\partial\, {}^{l}u_k(X_\alpha, 0)}{\partial X_\alpha}\delta_{3\,(i}\delta_{k)\,\alpha}\right\}\right],\\
&= 2\nu\frac{\partial\, {}^{l}u_k(X_\alpha, 0)}{\partial X_\alpha}\delta_{3\,(i}\delta_{k)\,\alpha} + \int_{\partial D} K_{ij}(X_\alpha, X'_\alpha)\, {}^{l}\check{\tilde{u}}_j(X_\alpha, X'_\alpha)\, dA(X'_\alpha),
\end{aligned}
$$

where

$$
K_{ij}(X_\alpha, X'_\alpha) \equiv \frac{1}{2\pi}\left(\frac{\delta_{ij}}{r_0^3} + \nu\,\delta_{i\beta}\cdot\frac{\partial^2\left(\frac{1}{r_0}\right)}{\partial X_\beta \partial X_\alpha}\cdot\delta_{\alpha j}\right)
$$

In deriving (4.16) from (4.15), we interchanged the limit and integration, which is permissible since the singularity of the integrand has been reduced to $1/r$. With the definitions

$$
\begin{aligned}
E_{i3j}(\bar{X}) &\equiv -\int_{\partial D - C} K_{ij}(\bar{X}, X')\, dA(X'), \\
G_{i3j\alpha}(\bar{X}) &\equiv 2\nu\,\delta_{3(i}\,\delta_{j)\alpha} - \int_{\partial D - C} (\bar{X}_\alpha - X'_\alpha)\, K_{ij}(\bar{X}, X')\, dA(X'),
\end{aligned}
\tag{4.17}
$$

we can write (4.16) as

$$
{}^{l}u_{i,3}(\bar{X}) = E_{i3k}\,{}^{l}u_k + G_{i3k\alpha}\,{}^{l}u_{k,\alpha} + \int_C K_{ij}(\bar{X}, X')\,{}^{l}\check{u}_j(\bar{X}, X')\, dA(X') . \tag{4.18}
$$

We remark that both E_{i3k} and $G_{i3k\alpha}$ are well defined at all interior points of C*. Remembering that an outer unit normal to ∂D has components $(0, 0, -1)$ from (4.6), $(4.10)_2$ and (4.18) we obtain (4.19) for surface tractions on C:

$$
\begin{aligned}
{}^{l}f_\alpha(\bar{X}) &= -\mu\left[{}^{l}u_{3,\alpha} + E_{\alpha 3k}\,{}^{l}u_k + G_{\alpha 3k\beta}\,{}^{l}u_{k,\beta}\right] - \mu \int_C K_{\alpha j}(\bar{X}, X')\,{}^{l}\check{u}_j(\bar{X}, X')\, dA(X'), \\
{}^{l}f_3(\bar{X}) &= -(\lambda + 2\mu)\left[E_{33k}\,{}^{l}u_k + G_{33k\alpha}\,{}^{l}u_{k,\alpha}\right] - \lambda \int_C \frac{{}^{l}\check{u}_3(\bar{X}, X')}{r_0^3}\, dA(X').
\end{aligned}
\tag{4.19}
$$

These relations make plain that the functional dependence of surface tractions on surface displacements is non-local. Also the functional dependence is not regular in the sense of VOLTERRA**. Moreover, in (4.19), one can integrate by parts and obtain a somewhat smoother kernel, implying thereby that the representation (4.19) is not unique. Depending upon the smoothness of the displacement field ${}^{l}\boldsymbol{u}$, this process would introduce line integrals along ∂C. For example, in $(4.19)_1$, when the term involving second derivatives of $1/r_0$ is integrated by parts, we obtain

$$
\begin{aligned}
{}^{l}f_\alpha(\bar{X}) = &-\mu\left[{}^{l}u_{3,\alpha} + E_{\alpha 3k}\,{}^{l}u_k + G_{\alpha 3k\beta}\,{}^{l}u_{k,\beta}\right] \\
&\frac{\mu}{2\pi} \int_C \left[\frac{{}^{l}\check{u}_\alpha(\bar{X}, X')}{r_0^3} + \nu \frac{\partial\left(\frac{1}{r_0}\right)}{\partial X_\beta} \frac{\partial\,{}^{l}\check{u}_\beta(\bar{X}, X')}{\partial X'_\alpha}\right] \\
&+ \frac{\mu}{2\pi}\,\varepsilon_{\alpha\gamma} \oint_{\partial C} \frac{\partial\left(\frac{1}{r_0}\right)}{\partial X_\beta}\,{}^{l}\check{u}_\beta(\bar{X}, X') \frac{dX'_\gamma}{ds}\, ds.
\end{aligned}
$$

The line integral represents the contribution to the surface tractions because of the edge effects and is well defined only at points away from the edge. If ${}^{l}\check{\boldsymbol{u}}$ is continuous across ∂C, the line integral would vanish in (4.20).

Substitution from $(2.29)_{1,2}$ into (4.19) yields the following boundary condition, for the part C of the boundary of B:

$$
\begin{aligned}
{}^{b}f_\alpha(\bar{X}) &= \mu\left[{}^{b}u_{3,\alpha} + E_{\alpha 3k}\,{}^{b}u_k + G_{\alpha 3k\beta}\,{}^{b}u_{k,\beta}\right] \\
&\quad + \frac{\mu}{2\pi} \int_C K_{\alpha j}(\bar{X}, X')\,{}^{b}\check{u}_j(\bar{X}, X')\, dA(X'), \\
{}^{b}f_3(\bar{X}) &= (\lambda + 2\mu)\left[E_{33k}\,{}^{b}u_k + G_{33k\alpha}\,{}^{o}u_{k,\alpha}\right] \\
&\quad + \frac{\lambda}{2\pi} \int_C \frac{1}{r_0^3}\,{}^{b}\check{u}_3(\bar{X}, X')\, dA(X').
\end{aligned}
\tag{4.21}
$$

---

* For some infinite regions, $G_{i3k\alpha}$ may not be well defined.

** See VOLTERRA [16, p. 20]. This is true of the functional dependence of ${}^{l}q$ on ${}^{l}v$, too *cf.* (3.20).

The derived boundary conditions (4.21) show that, when the body is loaded by a deformable loading device, the mechanical boundary conditions are neither of traction nor of place. The boundary conditions (4.21) do not support the viewpoint that B interacts with its surroundings locally, as is usually presumed in the consideration of follower forces. Also, (4.21) implies that the surface tractions depend upon the surface values ${}^b\boldsymbol{u}$ of $\boldsymbol{u}$ and not on the inward normal derivative of $\boldsymbol{u}$. In (4.21), λ and μ are elastic constants for the loading device and not for the body. In fact, no constitutive assumption was made for B.

We now elaborate the identity problem stated at the end of Section 2. The equation (4.1) is equivalent to

$$T_{i\alpha,\alpha}+T_{i3,3}=0. \tag{4.22}$$

Therefore the solutions (4.4) and (4.11) would satisfy (4.22) identically. Integrating (4.22) along the X_3-axes, using (4.6) and the fact that the stress vanishes at ∞, we see that the 2-dimensional balance of linear momentum, *viz*

$$\Phi^l_{i\alpha,\alpha}+{}^lf_i=0,$$

is also satisfied identically. Similar reasoning applies to the thermal problem of Section 3.

In the above example, the loading device was assumed to be a semi-infinite body. The other extreme case would be to consider a very thin body, *e.g.* the platings in MINDLIN'S [9] work or a thin shell as a loading device. In this case, with conventional approximations, the constitutive equations would be local.

In the example discussed above, infinitesimal deformations of D were considered and the governing equations were linear. This made the problem tractable. Moreover, the uniqueness of solutions or trivial non-uniqueness is well understood from the existing theorems. If we were to infer a functional similar to (4.19) from non-linear elasticity theory, we should encounter difficulties in principle. By mplication, the $\left\{\begin{matrix}\text{forces}\\ \text{displacements}\end{matrix}\right\}$ acting on C should determine the $\left\{\begin{matrix}\text{displacements}\\ \text{forces}\end{matrix}\right\}$ essentially uniquely. In non-linear elasticity, such a functional relationship would in general be non-unique. Possibly, adding some suitable side condition, say a minimum energy condition, would help alleviate the problem. In any event, mechanistic calculations of surface constitutive equations would be formidable.

We now give a few examples to illustrate that, in the nonlinear theory, the surface displacements may not uniquely determine the surface tractions. These examples concern non-linear *elastic intimate-contact mechanical loading devices* for which the surface tractions depend only on the present value of the deformation of the boundary points. It may be remarked that the fields of body force in B and L and of surface tractions on the part $\partial B - C$ of the boundary of the body are at our disposal to realize the various deformation fields contemplated in these examples.

Consider a horseshoe shaped body with the inner surface of its terminal ends pasted to a rod or a beam which acts as a loading device. Depending upon the initial deformation of the rod, it may buckle during subsequent deformations, thus resulting in non-unique surface tractions at the glued ends. Now conceive a thin hollow rubber hemisphere with its edges cemented to the bounding surface of the body, the former acting as a loading device for the latter. If it is possible to push in the hemisphere without displacing the glued edge, then in the deformed case the surface tractions on C would be quite different from those in the case when L is not distorted. Here the free surface F of L would be traction free in either case, except that internal stresses would be altered because of the change of shape. Another example is provided by the case when a rubber socket mounted on a cylinder acts as a loading device for the latter. Holding the outer surface of the socket fixed, we may rotate the cylinder through 360° so that the points on the common interface would occupy the initial position, but the surface tractions between the two would not be the same. For these examples, the additional requirement of minimum energy might sometimes uniquely determine the functional relationship between the surface tractions and the surface displacements.

5. Remarks

It should become clear from the preceding calculations that the functional representation for constitutive quantities is smoother when $\boldsymbol{f}$ and q are taken as the independent variables than when $\boldsymbol{u}$ and θ are so taken. In both cases, the kernel is of the convolution type. The kernel has an integrable singularity when $\boldsymbol{f}$ and q are taken as the independent variables. Because of the symmetry of the kernel, when $\boldsymbol{f}$ and q are sufficiently smooth a chain rule of differentiation holds. When $\boldsymbol{u}$ and θ are taken as the independent variables, the linear representations can be modified so as to reduce the singularity of the kernel to an integrable singularity.

A common feature of the 2-dimensional constitutive equations discussed in the previous two sections is that these are non-local. For a fixed $\overline{X}\in C$, the influence on constitutive quantities at $\overline{X}$ of q or of v at other points dies out as $\exp(-r^2)$ in the thermal problem; the corresponding rate of decay in the mechanical problem is just $1/r^2$ or $1/r$. Thus it seems advisable not to approximate the non-local terms by the local deformation, at least in the mechanical problem.

The expressions (3.6), (3.7) and (3.8) show that the linear heat conductor exhibits fading memory in the sense of COLEMAN & NOLL, even in the present 2-dimensional formulation. Indeed, to be specific, let us consider (3.8). Let us visualize two histories of heat flux defined on C such that these differ by a finite amount in the time interval $[0, t_0]$ but are identical afterwards. Then for t larger than t_0,

$$\begin{aligned}|\varepsilon_1(\overline{X},t)-\varepsilon_2(\overline{X},t)| &\leq \frac{c^2}{4\pi\kappa}\int_0^{t_0} d\tau \int_C H_2(\overline{X},X',t,\tau) \\ &\quad\times |{}^l q_1(X',\tau)-{}^l q_2(X',\tau)|\, dA(X'), \\ &\leq \frac{c^2}{4\pi\kappa}\,\frac{t_0}{(t-t_0)} \sup_{\substack{X'\in C\\ \tau\in[0,t_0]}} |{}^l q_1(X',\tau)-{}^l q_2(X',\tau)| \\ &\quad\times \int_C e^{\frac{-c r_0^2}{4\kappa t}}\, dA(X'),\end{aligned} \tag{5.1}$$

and therefore

$$\varepsilon_1(\bar{X}, t) = \varepsilon_2(\bar{X}, t) + O\left(\frac{1}{t}\right) \qquad \text{as } t \to \infty$$

Whereas it is usually assumed that the material has a fading memory, here for the linearly heat conducting half space, it has been shown to be so.

In the study of the thermal problem in Section 3, homogeneous initial conditions are assumed. Should we consider non-homogeneous initial data, we should find how the derived linear representations for the constitutive quantities depend upon the initial conditions. The results for the half space can be derived by considering instead of (3.4), a solution with non-homogeneous initial temperature distribution. Actually, it can be shown that, for a linear heat conductor, the memory of the initial conditions fades away exponentially with the passage of time. That is, the history of the boundary conditions determines, essentially uniquely, the temperature distribution and the stress field.

It seems that for a linear theory one can infer quite a bit about the general behavior of these surface constitutive functionals for more general shapes of bodies and more general material symmetries. However, for a first study we have treated a case where these functionals can be calculated quite explicitly.

For nonlinear loading devices, the formidable difficulties presented by the mechanistic derivation of surface constitutive equations suggest that a direct approach should be followed. The surface constitutive relations would, in general, be non-local in the sense that these would involve long range interaction. The deduced transformation laws exemplified by (2.95) show that it is safe to assume these surface constitutive relations to be invariant under the Galilean group of transformations. The constitutive equations of intimate-contact loading devices are also restricted by the requirement (2.31). If we adopt the viewpoint that the entropy inequality $(2.19)_5$ should hold for all solutions of the field equations (2.22) and $(2.24)_1$, then, for intimate-contact loading devices, $(2.19)_5$ should also be satisfied identically. In general, this may not give any additional restrictions on the constitutive relations for $\bar{x}_i$, $\Phi^l_{i\alpha}$, *etc.*, since in $(2.19)_5$, η, J_a and h are surface constitutive quantities. However, when either

$$J_a^* = \frac{Q_a^*}{\theta}$$

or

$$h^* = \frac{r^*}{\theta},$$

then $(2.19)_5$ might also restrict the constitutive equations for $\bar{x}_i$, $\Phi^l_{i\alpha}$, *etc.*

Sometimes we need to minimise the total energy E, defined as

$$\begin{aligned} E &\equiv \int_{R_b} \varepsilon^* \, dV + \int_{R} \varepsilon^* \, dV, \\ &= \int_{R_b} \varepsilon^* \, dV + \int_{C} \varepsilon \, dA + \oint_{\partial C} \varepsilon_e \, ds, \end{aligned}$$

with respect to a suitable class of variations of the boundary data on ∂B. In these cases, we often need to complete the function space consisting of functions describing the variations of the boundary data on ∂B, with respect to a suitable norm. The linear representations derived in the previous two sections suggest that an appropriate norm for this purpose would be a supremum norm rather than the L^p norm. These representations also favor the use of the supremum norm in the discussion of the Fréchet differentiability of the surface constitutive functionals.

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