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On the Asymptotic Stability of the Rest State of a Viscous Fluid bounded by a Rigid Wall and an Elastic Membrane

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1. Introduction

Whenever there is an interaction between a dissipative body and a non-dissipative one, the usual expectation is that the energy of the non-dissipative body will also be dissipated. Furthermore, if the system consisting of the two is isolated in the sense that there is no mechanical work done upon it by external forces, one expects that the system will approach a state of rest. An example of such a system is an elastic body submerged in a viscous fluid. Another similar but easily manageable problem is the isolated system formed by a viscous fluid at rest in a container whose walls are made of a linear elastic membrane and a rigid undeformable material. I study the manageable problem and prove that the energy of arbitrary disturbances of the rest state eventually decays. I assume that no external force acts upon the system, that the fluid adheres to the walls, that the surface tractions are continuous across the common interface between the fluid and the membrane, and that the strain energy of the membrane is a non-negative homogeneous quadratic form in the displacement gradients and is invariant under rigid body motions. Further, once the system is given an initial disturbance, the rigid portion of the walls of the container is maintained at rest subsequently. I assume also the existence of a weak solution of the equations governing the deformation of the fluid and the membrane, and I do not obtain the rate of decay of the energy.

It is worth mentioning that even though I show that the strain energy of the membrane approaches the value it takes in the unperturbed configuration, the displacements of all points of the membrane, measured from the initial undisturbed configuration, need not go to 0 as $t \to \infty$. This is due to the assumption on the form of the strain energy of the membrane.

First I prove the result for an incompressible Navier-Stokes fluid; then I prove it for a Reiner-Rivlin fluid.

2. Formulation of the Problem

Assume that, in the unperturbed (reference) configuration, the fluid occupies a smooth connected and bounded region R with a smooth boundary ∂R. The surface formed by the walls of the vessel containing the fluid in the reference configuration is to be smooth enough (see CAMPANATO [1] and FICHERA [2]) to apply the divergence theorem, the Poincaré inequality, the Korn inequality and the theorem of trace. Let $\partial_1 R$ be the part of the walls made of a rigid material and let the remaining portion $\partial_2 R \equiv \partial R - \overline{\partial_1 R}$ of the walls be an elastic membrane. I assume that $\partial_2 R$ is smooth enough to apply the surface divergence theorem.

The position of a material particle in the reference configuration is denoted by X and its position at time t by $\boldsymbol{x}(X, t) \equiv \chi(X, t)$. Thus $\boldsymbol{u}(X, t) \equiv \boldsymbol{x} - X$, and $\boldsymbol{v}(X, t) \equiv \frac{\partial \chi}{\partial t}(X, t) \equiv \dot{\boldsymbol{x}}$ give, respectively, the displacement and the velocity of X at time t. The surface co-ordinates Z^α on ∂R are given by a smooth transformation $X = X(Z^\alpha)$. Hereafter, a comma followed by an index i (α) indicates partial differentiation with respect to the rectangular Cartesian coordinates x^i (Z^α). The indices i, j *etc.* (α, β *etc.*) range over 1, 2, 3 (1, 2).

When there are no external forces, the equations governing the mechanical deformations of the fluid are

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0 \quad \text{in } \chi(R, t), \\ \rho \dot{\boldsymbol{v}} &= \operatorname{div} \boldsymbol{T} \quad \text{in } \chi(R, t), \\ \boldsymbol{T}\boldsymbol{N} &= -\boldsymbol{f} \quad \text{on } \chi(\partial_2 R, t), \\ \boldsymbol{v} &= \boldsymbol{0} \quad \text{on } \chi(\partial_1 R, t), \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{T} &= -p\boldsymbol{1} + 2\mu \boldsymbol{d}, \\ d_{ij} &= v_{(i,j)} \equiv \tfrac{1}{2}(v_{i,j} + v_{j,i}), \\ N_i &= \varepsilon_{ijk} \frac{\partial x^j}{\partial Z^1} \frac{\partial x^k}{\partial Z^2}. \end{aligned}$$

The deformations of the membrane are governed by the equations

$$\begin{aligned} \rho_m \dot{\boldsymbol{v}} &= \left(\frac{\partial W}{\partial \boldsymbol{u}_{,\alpha}} \right)_{,\alpha} + \boldsymbol{f} \quad \text{on } \chi(\partial_2 R, t), \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial \chi(\partial_2 R, t). \end{aligned}$$

Here $\boldsymbol{T}$ is the Cauchy stress tensor, $\boldsymbol{d}$ is the strain-rate tensor and ε_{ijk} is the alternating tensor having the value 1 or -1 if i, j, k form an even or odd permutation, respectively, of 1, 2, 3 and vanishing otherwise. Summation over repeated indices is implied. Further, ρ and μ denote, respectively, the mass density per unit volume and the shear viscosity of the fluid, $p(\boldsymbol{x}, t)$ gives the arbitrary hydrostatic pressure; $\boldsymbol{f}$ denotes the surface tractions per unit coordinate area $dZ^1\, dZ^2$ exerted by the fluid on the membrane, ρ_m is the mass density and W the strain energy of the membrane, each measured per unit coordinate area $dZ^1\, dZ^2$. W is assumed to be a non-negative homogeneous quadratic form in $\boldsymbol{u}_{,\alpha}$ and is normalized to take the value 0 in the reference configuration. Note that for the same system, different choices of the surface coordinates will, in general, result in different values of ρ_m and $\boldsymbol{f}$. The vector N defined by $(2.2)_3$ points along the outward normal to the surface in the present configuration and need not be of unit magnitude. Use of it rather than of the usual unit normal vector permits partial differentiation in $(2.3)_1$. In order that the assumptions that the fluid be incompressible and it adhere to the walls be mutually consistent, the deformations of the membrane must be such as to leave the total volume of the fluid invariant. Henceforth, I assume that the fluid is homogeneous. The analysis can be easily modified to apply to an inhomogeneous fluid [3]. Thus both the density and the shear viscosity are constant throughout the fluid, and I take them to be positive. For use in the definition of a weak solution I set

$$\begin{aligned}\Phi=\{\boldsymbol{\phi}\mid\boldsymbol{\phi}:\overline{\chi(R,t)}\to E^3,\ &\text{and for every } t>0,\ \boldsymbol{\phi}\in C^1(\overline{\chi(R,t)}\times(0,t)),\\ \boldsymbol{\phi}=\mathbf{0}\ \text{on}\ \chi(\partial_1 R,t),\ &\int|\psi_1|^2\,dV\leqq K_1,\ \int|\psi_2|^2\,dA\leqq K_2,\\ \text{for } \psi_1=\boldsymbol{\phi},\dot{\boldsymbol{\phi}},\boldsymbol{\phi}_{,i}\ &\text{and}\ \psi_2=\boldsymbol{\phi},\dot{\boldsymbol{\phi}},\boldsymbol{\phi}_{,\alpha}\}.\end{aligned} \tag{2.4}$$

Here E^3 denotes the usual 3-dimensional Euclidean space, K_1 and K_2 are positive constants, and the volume integration signified by the presence of the volume measure dV under the integral sign is over $\chi(R,t)$. Also the surface integration signified by the presence of the area measure $dA\equiv dZ^1\,dZ^2$ under the integral sign is over $\chi(\partial_2 R,t)$. Taking the inner product of $(2.1)_2$ and $(2.3)_1$ with $\boldsymbol{\phi}$, integrating the resulting equations over $\chi(R,t)$ and $\chi(\partial_2 R,t)$, adding these two equations and simplifying by using the divergence theorem, the surface divergence theorem and the boundary conditions $(2.1)_{3,4}$ and $(2.3)_2$ yields

$$\begin{aligned}&\frac{d}{dt}\left[\int\rho\mathbf{v}\cdot\boldsymbol{\phi}\,dV+\int\rho_m\mathbf{v}\cdot\boldsymbol{\phi}\,dA\right]-\left[\int\rho\mathbf{v}\cdot\dot{\boldsymbol{\phi}}\,dV+\int\rho_m\mathbf{v}\cdot\dot{\boldsymbol{\phi}}\,dA\right]\\ &\quad=\int p\phi_{i,i}\,dV-2\mu\int d_{ij}\phi_{i,j}\,dV-\int\frac{\partial W}{\partial u^i_{,\alpha}}\phi^i_{,\alpha}\,dA.\end{aligned}$$

For use in the proof of the theorem stated below, I briefly recall the inequalities due to POINCARÉ and KORN, and the theorem of trace. For a differentiable vector-valued function $\boldsymbol{f}$ defined on a smooth domain D such that $\boldsymbol{f}\in L^2(D)$, $f_{i,j}\in L^2(D)$ and $\boldsymbol{f}=\mathbf{0}$ on a part $\partial_0 D$ of the boundary ∂D of D, POINCARÉ's inequality [CAMPANATO, 1]

$$\int_D f^2\,dV\leqq p_1\int_D f_{i,j}f_{i,j}\,dV,$$

and KORN's inequality [CAMPANATO, 1]

$$\int_D f_{i,j}f_{i,j}\,dV\leqq p_2\int_D f_{(i,j)}f_{(i,j)}\,dV$$

hold. The constants p_1 and p_2 appearing in these inequalities depend upon D and $\partial_0 D$. For a differentiable vector-valued function $\boldsymbol{f}$ defined on the closure $\bar{D}$ of a properly regular domain D such that $\boldsymbol{f}\in L^2(\bar{D})$, $f_{i,j}\in L^2(\bar{D})$ and the restriction of $\boldsymbol{f}$ to ∂D is square-integrable over ∂D, the theorem of trace states that [FICHERA, 2]

$$\int_{\partial D} f^2\,dA\leqq p_3\left[\int_D f^2\,dV+\int_D f_{i,j}f_{i,j}\,dV\right] \tag{2.8}$$

where p_3 is a function of the domain D.

Since the two-dimensional measure of $\chi(\partial_1 R, t)$ is time-independent, $\boldsymbol{v}$ satisfies the side condition sufficient for (2.6) and (2.7) to hold provided the two-dimensional measure of $\partial_1 R$ is positive. This last condition is satisfied by virtue of the assumptions that a part of the walls of the container is made of a rigid material and the fluid adheres to the walls. Thus if for every $t>0$, $\boldsymbol{v}\in L^2(\overline{\chi(R, t)})$, $v_{i,j}\in L^2(\overline{\chi(R, t)})$, $\boldsymbol{v}\in L^2(\chi(\partial_2 R, t))$, then from (2.6), (2.7) and (2.8)

$$\int v^2\,dV \leqq p_1 \int v_{i,j} v_{i,j}\,dV, \tag{2.9}$$

$$\int v_{i,j} v_{i,j}\,dV \leqq p_2 \int d_{ij} d_{ij}\,dV, \tag{2.10}$$

$$\int v^2\,dA \leqq p_3 \left[\int v^2\,dV + \int v_{i,j} v_{i,j}\,dV\right]. \tag{2.11}$$

Here, p_1 and p_2 depend upon $\chi(R, t)$ and $\chi(\partial_1 R, t)$ whereas p_3 is a function of $\chi(R, t)$. Since the region of integration varies with time t, it follows that p_1, p_2 and p_3 also vary with t and hence are real-valued functions of t.

I now introduce a definition of the weak solution. A *weak solution* of (2.1)–(2.3) is a pressure field p and a mapping $\chi : \bar{R} \times (0, t) \to E^3$ such that for every $t>0$

(i) χ is twice continuously differentiable with respect to X and t in $\bar{R}\times(0, t)$,

(ii) $\det\left|\dfrac{\partial\chi}{\partial X}\right| = 1\ \forall X\in R;\quad \boldsymbol{u}, \boldsymbol{v}\in\Phi$,

(iii) $\sup\limits_{\substack{X\in R\\ t>0}} |p|$ is finite,

(iv) $P_n \equiv \sup\limits_{t>0} p_n(t),\quad n=1, 2, 3$ is finite,

(v) (2.5) is satisfied for every $\phi\in\Phi$, and

(vi) $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfy initial conditions.

Note that $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfy kinematic boundary conditions because of (ii) above and the definition of Φ. I assume that the set S of initial conditions defined below by (2.12) is non-empty.

$S \equiv \{(\boldsymbol{u}_0, \boldsymbol{v}_0) : \boldsymbol{u}_0(\cdot) \equiv \boldsymbol{u}(\cdot, 0),\ \boldsymbol{v}_0(\cdot) \equiv \boldsymbol{v}(\cdot, 0),\ E(0) \leqq$ constant, and there exists a weak solution of (2.1)–(2.3) for every $t>0$ satisfying these initial conditions$\}$. (2.12)

$$E(t) \equiv \int W(\cdot, t)\,dA + \frac{\rho}{2}\int v^2(\cdot, t)\,dV + \tfrac{1}{2}\int \rho_m v^2(\cdot, t)\,dA \tag{2.13}$$

is the total energy of the system at time t. The main result is the following

Theorem. *For every initial disturbance in S, the weak solution exhibits the behavior*

$$\int v^2\,dV \to 0, \quad \int v^2\,dA \to 0, \quad \int W\,dA \to 0 \quad \text{as } t\to\infty \tag{2.14}$$

provided

$$\inf_{Z\in\partial_2 R} \rho_m > 0, \qquad \sup_{Z\in\partial_2 R} \rho_m \leqq \text{constant}, \tag{2.15}$$

and the 2-dimensional measure of $\partial_1 R$ is positive.

3. Proof of the Theorem

Since $\boldsymbol{v}\in\Phi$, take $\boldsymbol{\phi}=\boldsymbol{v}$ as the test function. With this choice of the test function and by use of $(2.1)_1$, (2.5) is simplified to

$$\dot{E}=-2\mu\int d_{ij}d_{ij}\,dV.$$

From (2.13) and (3.1), conclude that $E\geqq 0$, $-K_1\leqq\dot{E}\leqq 0$ and

$$\lim_{t\to\infty} E(t) \text{ exists.}$$

Also, for every $t>0$

$$E(t)\leqq E(0),$$

and, in particular,

$$w(t)\equiv\int W(\cdot,t)\,dA\leqq E(0). \tag{3.3}$$

Thus the total strain energy of the membrane stays bounded. Since $\dot{E}$ is bounded, $E(t)$ is of bounded variation on $(0, T)$, T being an arbitrary real positive number. Integrating (3.1) over $(0, T)$ and recalling that $E(t)\geqq 0$ yields

$$E(0)\geqq 2\mu\int_0^T dt\int d_{ij}d_{ij}\,dV, \tag{3.4}$$

and hence

$$\int d_{ij}d_{ij}\,dV\in L^1(0,\infty). \tag{3.5}$$

It is clear from the definition of the weak solution that the inequalities (2.9)–(2.11) hold and are stregthened when p_1, p_2 and p_3 are replaced, respectively, by P_1, P_2 and P_3. From strengthened versions of (2.9) and (2.10), I conclude that $\boldsymbol{v}$ satisfies

$$\int v^2\,dV\leqq P_1P_2\int d_{ij}d_{ij}\,dV \tag{3.6}$$

and this with (3.5) leads to the conclusion that

$$\int v^2\,dV\in L^1(0,\infty). \tag{3.7}$$

Now (3.7), (3.5) and the strengthened form of (2.11) give the following:

$$\int v^2\,dA\in L^1(0,\infty). \tag{3.8}$$

If we recall that ρ is finite and ρ_m is bounded, from (3.7) and (3.8) it follows that

$$\rho\int v^2\,dV+\int\rho_m v^2\,dA\in L^1(0,\infty).$$

Rewriting (3.1) as

$$\frac{d}{dt}\left[\rho\int v^2\,dV+\int\rho_m v^2\,dA\right]=-\dot{w}-2\mu\int d_{ij}d_{ij}\,dV,$$

and recalling the definition of the weak solution, I conclude that $\dot{w}$ is bounded, and hence w is of bounded variation on $(0, T)$. This implies that $\dot{w}\in L^1(0, T)$. Use of (3.3) shows that

$$\left|\int_0^T\dot{w}(t)\,dt\right|\leqq 2E(0),$$

and since the left-hand side of (3.11) is independent of T, it follows that $\dot{w}\in L^1(0, \infty)$. From (3.10) and (3.5) I obtain

$$\frac{d}{dt}\left[\rho\int v^2\,dV+\int \rho_m v^2\,dA\right]\in L^1(0,\infty). \tag{3.12}$$

Thus $\rho\int v^2\,dV+\int\rho_m v^2\,dA$ is uniformly continuous in t and

$$\rho\int v^2\,dV+\int\rho_m v^2\,dA\to 0 \quad \text{as } t\to\infty \tag{3.13}$$

now follows from (3.9). $(2.14)_1$ and $(2.14)_2$ are immediate consequences of (3.13), $(2.15)_1$ and the assumption that $\rho>0$.

From (3.2) and (3.13) it follows that

$$\lim_{t\to\infty}\int W(\cdot,t)\,dA \quad \text{exists.} \tag{3.14}$$

To prove $(2.14)_3$, it should now suffice to show that

$$\int W(\cdot,t)\,dA\in L^1(0,\infty). \tag{3.15}$$

To this end, I choose a test function $\phi\in\Phi$ such that

$$\phi(x,t)=u(x,t),\ (x,t)\in\chi(\partial R,t)\times(0,t) \tag{3.16}$$

and ϕ is, for the time being, unrestricted in the interior. For this choice of ϕ, equation (2.5) when integrated over $(0, T)$ yields

$$\begin{aligned}2\int_0^T w(t)\,dt= &-\rho\int v\cdot\phi\,dV\big|_0^T-\int\rho_m v\cdot u\,dA\big|_0^T+\int_0^T dt\int\rho_m v^2\,dA\\ &+\rho\int_0^T dt\int v\cdot\dot{\phi}\,dV+\int_0^T dt\int p\phi_{i,i}\,dV-2\mu\int_0^T dt\int d_{ij}\phi_{i,j}\,dV.\end{aligned} \tag{3.17}$$

I now try to find sufficient conditions on ϕ which suffice to conclude that each term on the right-hand side is bounded for arbitrary large T. The third term on the right-hand side of (3.17) is bounded because of (3.8) and $(2.15)_2$, and for the remaining terms, by using the Cauchy-Schwarz inequality wherever necessary, I obtain

$$\begin{aligned}&\left|\int v(\cdot,T)\cdot\phi(\cdot,T)\,dV\right|\leqq\left(\int v^2(\cdot,T)\,dV\right)^{\frac12}\left(\int\phi^2(\cdot,T)\,dV\right)^{\frac12},\\ &\left|\int\rho_m v(\cdot,T)\cdot u(\cdot,T)\,dA\right|\leqq\sup(\rho_m|u|)\int|v(\cdot,T)|\,dA,\\ &\left|\int_0^T dt\int v\cdot\dot{\phi}\,dV\right|\leqq\left(\int_0^T dt\int v^2\,dV\right)^{\frac12}\left(\int_0^T dt\int\dot{\phi}^2\,dV\right)^{\frac12},\\ &\left|\int_0^T dt\int p\phi_{i,i}\,dV\right|\leqq\sup|p|\int_0^T dt\int|\phi_{i,i}|\,dV,\\ &\left|\int_0^T dt\int d_{ij}\phi_{i,j}\,dV\right|\leqq\left(\int_0^T dt\int d_{ij}d_{ij}\,dV\right)^{\frac12}\left(\int_0^T dt\int\phi_{i,j}\phi_{i,j}\,dV\right)^{\frac12}.\end{aligned} \tag{3.18}$$

That the right-hand sides of $(3.18)_1$ and $(3.18)_2$ are bounded follows from $(2.15)_2$, and the definitions of the weak solution and of the set Φ. Thus if $\phi\in\Phi$ is such that ϕ satisfies (3.16) and the conditions

$$\int \dot{\phi}^2 \, dV \in L^1(0, \infty),$$

$$\int |\phi_{i,i}| \, dV \in L^1(0, \infty), \tag{3.19}$$

and

$$\int \phi_{i,j} \phi_{i,j} \, dV \in L^1(0, \infty),$$

I can conclude (3.15) and hence $(2.14)_3$. To demonstrate the existence of such a function, I introduce a new coordinate system in the neighborhood

$$R_\varepsilon \equiv \{X: X \in R, \text{ distance } (X, \partial R) \leqq \varepsilon\}$$

of ∂R, defined by a smooth transformation

$$X = X(Z^\alpha) - \eta \frac{N}{|N|}, \quad 0 \leqq \eta \leqq \varepsilon.$$

For sufficiently small ε, the transformation (3.20) between (X^1, X^2, X^3) and (Z^1, Z^2, η) is one-to-one. Define

$$\begin{aligned} \phi(x(X, t), t) &= \left(1 - \frac{\eta}{\varepsilon}\right) e^{-\eta t} u(x(X(Z^\alpha), t), t), \quad 0 \leqq \eta \leqq \varepsilon, \\ &= \mathbf{0} \quad \text{if } x \in \chi(R - R_\varepsilon, t). \end{aligned} \tag{3.21}$$

Recalling the definition of the set Φ, I note that ϕ defined by (3.21) is in Φ. Further this function satisfies (3.16) and (3.19). Thus the theorem is proved.

Since $v \in \Phi$, $\sup_{X \in R} |\dot{v}|$ is finite and therefore v is uniformly continuous in t. Now using also $(2.14)_{1,2}$, I conclude that the velocity of every particle of the fluid and the membrane approaches 0 as $t \to \infty$.

4. Reiner-Rivlin Fluids

I now indicate the modifications necessary for the above analysis to apply to the case when the viscous fluid is a Reiner-Rivlin fluid. The constitutive equation for a Reiner-Rivlin fluid is

$$T = -p\mathbf{1} + f_1 d + f_2 d^2 \tag{4.1}$$

where f_1 and f_2 are arbitrary functions of the second and third invariants of d, which are denoted by II and III. I assume that f_1 and f_2 are bounded. If I proceeded as I did for the Navier-Stokes fluid, I should obtain, instead of (2.5), the following:

$$\begin{aligned} &\frac{d}{dt}\left[\rho \int v \cdot \phi \, dV + \int \rho_m v \cdot \phi \, dA\right] - \left[\rho \int v \cdot \dot{\phi} \, dV + \int \rho_m v \cdot \dot{\phi} \, dA\right] \\ &= \int p \phi_{i,i} \, dV - \int (f_1 d_{ij} \phi_{i,j} + f_2 d_{ij}^2 \phi_{i,j}) \, dV - \int \frac{\partial W}{\partial u^i_{,\alpha}} \phi^i_{,\alpha} \, dA. \end{aligned} \tag{4.2}$$

I modify the definition of the weak solution so as to require also that $v_{i,j} v_{j,k} \in L^2(\chi(R, t))$. Setting $\phi = v$ in (4.2), using also the Hamilton-Cayley theorem*

* See [3, section 4].

to simplify the right-hand side, I obtain

$$\dot{E} = -\int \{(f_1 + \mathrm{II} f_2 - f_2) \operatorname{tr} \boldsymbol{d}^2 + f_2 \operatorname{tr}(\boldsymbol{d}^2 + \boldsymbol{d}^3 + \boldsymbol{d}^4)\} \, dV.$$

Since

$$\operatorname{tr}(\boldsymbol{d}^2 + \boldsymbol{d}^3 + \boldsymbol{d}^4) \geqq 0 \, \forall \boldsymbol{d},$$

the inequalities

$$f_1 - f_2 + \mathrm{II} f_2 \geqq \text{const} > 0, \quad f_2 \geqq 0 \ \forall \boldsymbol{d} \tag{4.3}$$

are sufficient to conclude $(2.14)_{1,2}$ for weak solutions in the case of Reiner-Rivlin fluids. Proceeding as I did before to arrive at (3.17), I now obtain an expression in which the last term $2\mu \int_0^T dt \int d_{ij} \phi_{i,j} dV$ on the right-hand side of (3.15) is replaced by

$$\int_0^T dt \int f_1 \, d_{ij} \phi_{i,j} \, dV + \int_0^T dt \int f_2 \, d_{ij}^2 \phi_{i,j} \, dV. \tag{4.4}$$

That the first term in (4.4) is bounded follows from (3.5), $(3.19)_3$ and the assumption that f_1 is bounded. As for the second term,

$$\begin{aligned}
\left| \int_0^T dt \int f_2 \, d_{ij}^2 \phi_{i,j} \, dV \right| &\leqq \sup |f_2| \int_0^T dt \int |d_{ij}^2 \phi_{i,j}| \, dV, \\
&\leqq \sup |f_2| \int_0^T dt \int |d_{ij}^2 d_{ij}^2|^{\frac{1}{2}} |\phi_{i,j} \phi_{i,j}|^{\frac{1}{2}} \, dV, \\
&\leqq \sup |f_2| \sup(\operatorname{tr} \boldsymbol{d}^2) \int_0^T dt \int |\phi_{i,j} \phi_{i,j}|^{\frac{1}{2}} \, dV.
\end{aligned}$$

I used the Cauchy-Schwarz inequality at the intermediate step and the relation* $\operatorname{tr} \boldsymbol{d}^4 = (\operatorname{tr} \boldsymbol{d}^2)^2$ to arrive at the final inequality. The relation (3.5) implies that

$$\sup |\operatorname{tr} \boldsymbol{d}^2| < \infty$$

and for the function $\boldsymbol{\phi}$ given by (3.21)

$$\int |\phi_{i,j} \phi_{i,j}|^{\frac{1}{2}} \, dV \in L^1(0, \infty).$$

Thus the corresponding theorem for a Reiner-Rivlin fluid follows. The inequalities (4.3) delimit the class of Reiner-Rivlin fluids to which the theorem applies. That the class is not empty becomes clear from the following example. If

$$f_2 = -\beta \mathrm{II},$$

$$f_1 = \alpha - \beta \mathrm{II} + \beta \mathrm{II}^2,$$

α and β positive constants, the inequalities (4.3) hold.

5. Remarks

So far I have proved that the total strain energy of the membrane goes to 0 ultimately. I now show that for the problem at hand this fact does not necessarily imply that the displacements also go to zero in L^2-norm.

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* *Cf.* [3, section 4].

A homogeneous quadratic form in $u_{,\alpha}$ is expressible as a homogeneous quadratic form in $\varepsilon_{\alpha\beta}$ and $\omega_{\alpha\beta}$, defined as follows:

$$\varepsilon_{\alpha\beta} \equiv \tfrac{1}{2}(\boldsymbol{u}_{,\alpha} \cdot \boldsymbol{X}_{,\beta} + \boldsymbol{u}_{,\beta} \cdot \boldsymbol{X}_{,\alpha}),$$
$$\omega_{\alpha\beta} \equiv \tfrac{1}{2}(\boldsymbol{u}_{,\alpha} \cdot \boldsymbol{X}_{,\beta} - \boldsymbol{u}_{,\beta} \cdot \boldsymbol{X}_{,\alpha}). \tag{5.1}$$

That is,

$$W(u_{,\alpha}) = \hat{W}(\varepsilon_{\alpha\beta}, \omega_{\alpha\beta}). \tag{5.2}$$

For a rigid motion of the membrane $\varepsilon_{\alpha\beta}=0$, and if W is to be invariant under rigid motions, $\hat{W}$ must not depend upon $\omega_{\alpha\beta}$. Thus

$$W(u_{,\alpha}) = \overline{W}(\varepsilon_{\alpha\beta}) \tag{5.3}$$

and since $W(0)=0$, $\overline{W}(0)=0$. For a plane membrane that occupies the surface $X_3=0$ in the reference configuration, the displacement field

$$u_1 = u_2 = 0, \qquad u_3 = \psi,$$

where ψ is any smooth function of Z^α and vanishes on the boundary of the membrane, satisfies (2.3) with $\boldsymbol{v}=\boldsymbol{f}=\boldsymbol{0}$. For displacements given by (5.4),

$$\varepsilon_{\alpha\beta}=0,$$

and, therefore,

$$W(u_{,\alpha})=0$$

but $\int u^2\,dA$ need not be zero. This example shows that the membrane may have several distinct rest states, not differing from each other by a rigid motion, in which the total strain energy of the membrane is same. What the above theorem gives is that the membrane returns to one state of rest.

However, for a plane membrane for which the strain energy satisfies the inequality

$$W dA \geqq \text{constant} \int \varepsilon_{\alpha\beta}\varepsilon_{\alpha\beta}\,dA,$$

one can easily show that the tangential displacements go to 0 in L^2-norm as $t \to \infty$. Indeed, setting $Z^\alpha = X^\alpha$ in $(5.1)_1$, and recalling Poincaré's inequality (2.6) and Korn's inequality (2.7), one obtains

$$\int u_\alpha u_\alpha\,dA \leqq \text{constant} \int W dA$$

and hence

$$\int u_\alpha u_\alpha\,dA \to 0 \quad \text{as } t \to \infty.$$

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