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IMPULSIVELY LOADED CIRCULAR PLATES

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Abstract—The dynamic behaviour of elastic-plastic circular plates, with deflections in the range where both bending moments and membrane forces are important, is investigated. The formulation is restricted to two dimensional and axisymmetric responses. The effect of shear deformations, rotary inertia and material strain rate sensitivity is not considered. The equations of motion are solved for small deformations from the initial flat configuration of the plate. The influence of the curvature of the deformed surface, on the ensuing deformations is considered by contemplating that the successive deformations take place w.r.t. the convected axes. Thus, a kinematically admissible velocity field compatible with the aforementioned solution is used for predicting the response of the plate to large deformations. The superposition of the successive increments in displacement and strain is carried out by referring each to the fixed global axes. Using this technique, the deformed shape of the plate and the initial velocity as a function of central deflection are computed and compared with the corresponding experimental findings.

NOTATION

a	radius of the plate
a_i^j	transformation coefficients
B_0, B	reference and current configuration
$de_{ij}, d\bar{e}_{ij}$	incremental strain tensors with respect to global and convected axes
$de_{rr}, de_{\theta\theta}, de_{zz}, de_{rz}$	radial, circumferential, thickness and transverse shear strain increments
dE_s	incremental strain energy of deformation
dKE	change in kinetic energy during small deformation
ds, dS	length of line element in the reference and deformed configuration
$du_i, d\bar{u}_i$	incremental displacement components with respect to global and deformed axes
du, dv, dw	radial, circumferential and axial incremental displacement components
dW	work done by the prescribed forces during the small deformation
$d\sigma_{ij}, ds_{ij}, d\tau_{ij}$	incremental true stress, nominal stress and Kirchoff stress tensor
$d\sigma_{rr}, d\sigma_{\theta\theta}, d\sigma_{zz}, d\sigma_{rz}$	radial, circumferential, axial and transverse shear stress increments
dp	increment in hydrostatic pressure
E	Young's modulus of plate material
g_{ij}, G_{ij}	metric tensors in reference and current configurations
H	strain hardening parameter
h	half the plate thickness
K_{ijkl}	elastic moduli
k_1, k_2	direction ratios of local normal to the yield surface
m_{ij}	components of unit normal to the yield surface
r, θ, z	co-ordinate of a point with respect to cylindrical axes
S	surface area
t	time
V	volume enclosed by surface S
V_0	initial transverse velocity
x_i, X_i	curvilinear co-ordinates of the point in reference and current states
$\overline{X}_i$	coordinates with respect to convected axes
β, ξ	dimensionless variables r/a, z/a
λ, μ	Lamé constants
δ	$2\mu/H$
ρ	density of plate material

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σ	generalised stress
σ_0	yield stress in simple tension
Γ^i_{jk}	Christoffel symbols
W_0	total central deflection
W	total deflection at a point

1. INTRODUCTION

The study of the existing literature on the dynamic deformation of disks reveals that most efforts have been concentrated on investigating the plate response in which either the membrane forces [1–3] or the bending moments [4–6] have been considered important. Whereas both these approaches over-estimate the final deformations, the accuracy of the membrane solution improves with the increase of deflection and that of the bending deteriorates. Recently, it has been demonstrated [7–9] that a significant improvement on the prediction of the plate behaviour subjected to impulsive loading can be achieved if the combined effect of bending moments and membrane forces is included in the analysis. However, these approaches seem to be directed towards estimating the final central deflection rather than the deformed profile of the plate. The rigid perfectly plastic solution of Refs. [1, 2] considering membrane forces postulate a conical shape with a dimpling tendency (zero thickness) at the centre. The present investigation is aimed at predicting the ultimate deformed shape by considering elastic-plastic deformations and the combined effect of bending moments and membrane forces.

Both mathematical and physical considerations suggest the use of the incremental theory. The motivation to this effect is provided, in part, by the reasoning given in [16] that the total stress–strain laws have a limited applicability when considering work hardening of solids. Also, the use of incremental theory would help to account for the continuously varying curvature of the deformed surface. The difficulty encountered in the superposition of these small deformations is overcome by transferring each increment in displacement and strain to the global axes. This approach enables us to maintain the compatibility in the stress–strain relation by assuming that only the stress in the thickness direction to be zero rather than both the stress and strain to be zero.

2. FORMULATION OF THE INCREMENTAL PROBLEM

The strain increment during a small incremental deformation from some reference configuration B_0 of the body to the deformed state B is taken as

$$\mathrm{d}e_{ij} = \tfrac{1}{2}(G_{ij} - g_{ij}) \tag{2.1}$$

where

$$G_{ij} = a_i^R a_j^S g_{RS} \tag{2.2}$$

$$a_i^R = \frac{\partial X^R}{\partial x^i} \tag{2.3}$$

and g_{ij}, g_{RS} are the co-variant metric tensors with respect to the curvilinear axes (x_i) in B_0 in the reference and deformed states respectively and $\mathrm{d}u^i$ represents the displacement components in going from B_0 to B. To the first order of approximation, equation (2.1) is simplified [10] to

$$de_{ij} = \tfrac{1}{2}(du_{i,j}+du_{j,i}). \tag{2.4}$$

Here comma denotes co-variant differentiation with respect to the axes in B_0. If during the continuing deformations of the material body, the additional incremental displacements $d\bar{u}^i$ are measured with respect to the axes $\bar{X}_i$, embedded in the body and deformed by the whole preceding strain, the corresponding strain increments are expressed by a relation similar to equation (2.4). That is

$$d\bar{e}_{ij} = \tfrac{1}{2}(d\bar{u}_{i,j}+d\bar{u}_{j,i}) \tag{2.5}$$

where the differentiation is to be performed with respect to the convected axes. The following transformations

$$du^r = a^r_i\, d\bar{u}^i \tag{2.6}$$

$$d\bar{e}_{ij} = a^r_i a^s_j\, de_{rs} \tag{2.7}$$

where

$$a^r_i = \delta^r_i + \frac{\partial(du^r)}{\partial x^i} \tag{2.8}$$

enable us to find the resulting global components of the incremental displacements and strains. So far, no mention has been made regarding the variation in the forces (or the stresses) induced as a result of the deformation. The evaluation of these stress increments requires a complete knowledge of the previous stress distribution in the body. For instance, corresponding to the following stress state

$$\sigma^{ij} = s^{ij} = \tau^{ij} \tag{2.9}$$

in the current configuration, where σ^{ij}, s^{ij}, τ^{ij} are, respectively, the true stress, nominal stress and the Kirchoff stress tensors; the variation in the stresses because of the small deformation of the body are given by [11]

$$\dot{s}^{ij} = \dot{\tau}^{ij} - \sigma^{jk}\dot{u}^i{}_{,k} \tag{2.10}$$

and [12]

$$\dot{\sigma}^{ij} = \dot{\tau}^{ij} - \sigma^{ij}\dot{u}^k{}_{,k} \tag{2.11}$$

where dot stands for material derivative that is the time derivative following the element. The equation (2.10) is also the necessary and sufficient condition that the angular equation of equilibrium be satisfied. The equations of motion for the incremental deformation are

$$ds^{ij}{}_{,i} = \rho\, d\ddot{u}^j. \tag{2.12}$$

Also during the small incremental deformation, the work dW done on a volume V of the material medium by the prescribed loads on its surface S with local unit normal n_i, can be expressed as

$$dW = \int_S n_i(s^{ij}+\tfrac{1}{2}\, ds^{ij})\, du_j\, dS \tag{2.13}$$

$$= \int_V \rho(\ddot{u}^j+\tfrac{1}{2}\, d\ddot{u}^j)\, du_j\, dV + \int_V (s^{ij}+\tfrac{1}{2}\, ds^{ij})\, du_{j,i}\, dV. \tag{2.14}$$

In this simplification, Green's theorem and equation (2.12) have been used. The first integral on the right hand side represents the work done against the inertia forces and the second gives the increase in strain energy of deformation. Since the work done against the inertia forces is equal to the change $\mathrm{d}KE$ in kinetic energy, therefore,

$$\mathrm{d}W = \mathrm{d}KE + \mathrm{d}E_d. \tag{2.15}$$

An interesting case arises when an initial axisymmetric transverse velocity has been imparted to the entire body and no loads are prescribed on its surface except on the portion where zero displacements have been specified. For this type of loading

$$\mathrm{d}W = 0$$

and

$$\mathrm{d}E_d = -\mathrm{d}KE. \tag{2.16}$$

That is, the deformation would cease as soon as all the kinetic energy given to the system has been transformed into the strain energy.

3. MATERIAL PROPERTIES

In order to separate the physics of deformation from the changes in geometry, it is desirable to define a stress increment which is unaffected by the rigid body motion. The Jaumann derivative of the Kirchoff's stress (which is associated with axes embedded in the material but not deforming with it and vanishes in the absence of deformation) is suitable for defining the stress increments in the constitutive equation. The fixed components, $D\tau^{ij}/Dt$ of the Jaumann derivative of the Kirchoff stress as given by Oldroyd [13], are:

$$\frac{D\tau^{ij}}{Dt} = \dot{\tau}^{ij} + \sigma^{ik}\omega_k^j + \sigma^{jk}\omega_k^i \tag{3.1}$$

where $\omega_{ij} = \frac{1}{2}(\dot{u}_{i,j} - \dot{u}_{j,i})$ is the skewsymmetric velocity gradient tensor. The constitutive equation for an elastic-plastic solid is taken in the form [12]

$$\frac{D\tau^{ij}}{Dt} = K^{ijkl}(\dot{e}_{kl} - \dot{e}_{kl}^p) \tag{3.2}$$

with

$$\dot{e}_{ij}^p = \begin{cases} H^{-1} m_{ij}\left(m_{kl}\dfrac{D\tau^{kl}}{Dt}\right) & \text{when } m_{ij}\dfrac{D\tau^{ij}}{Dt} > 0 \\ 0 & < 0 \end{cases} \tag{3.3}$$

where

K_{ijkl} = elastic moduli in the current state having the symmetry property with respect to ij and kl, i and j, k and l

H = positive scalar measure of the current rate of work hardening

$\dot{e}_{ij}^p$ = plastic part of strain rate, and

m_{ij} = components of unit outward drawn normal to the local yield surface.

The linear stress strain relation for elastic solids, derived from equation (3.2) by making H infinitely large is

$$\frac{D\tau^{ij}}{Dt} = K^{ijkl}\dot{e}_{kl}. \tag{3.4}$$

In finding the stress increments during an incremental deformation, it seems appropriate to use approximations similar to that made in the previous section in order to simplify the expression for the strain increments. Therefore, suppressing the time element and combining equations (3.2) and (3.3), the stress increments are expressed as:

$$D\tau^{ij} = K^{ijkl}\,\mathrm{d}e_{kl} - K^{ijkl}m_{kl}\,\frac{m_{rs}K^{rspq}\,\mathrm{d}e_{pq}}{H + m_{rs}K^{rspq}m_{pq}}. \tag{3.5}$$

Using equations (3.1) and (2.11), the increments in true stresses during the small deformation can be determined from

$$\mathrm{d}\sigma^{ij} = D\tau^{ij} - \sigma^{ij}\,\mathrm{d}u^{k}{}_{,k} - \sigma^{ik}\,\mathrm{d}\omega^{j}_{k} - \sigma^{jk}\,\mathrm{d}\omega^{i}_{k}. \tag{3.6}$$

Since no assumption has been made regarding the form of K_{ijkl} or the yield surface, this formulation is applicable to a wide class of solids. However, for an isotropic elastic solid

$$K_{ijkl} = \lambda g_{ij}g_{kl} + \mu(g_{ik}g_{jl} + g_{il}g_{jk}) \tag{3.7}$$

where λ, μ are the Lamé constants. For an incompressible elastic–plastic solid

$$\mathrm{d}u^{k}{}_{,k} = 0 \tag{3.8}$$

and relation (3.6) gives the increment in deviatoric stresses.

4. ASSUMPTIONS

In the study of the dynamic behaviour of circular plates, following assumptions have been made or implied.

(i) The dynamic load is supposed to be such that at some given time, an axisymmetric velocity field is instantaneously imparted to the entire plate, save at the edges where the velocity is zero and thereafter the plate is subjected only to the edge forces.

(ii) The plate thickness is small as compared to its radius.

(iii) Linear elements initially normal to the middle surface preserve this normality with respect to the axes embedded in the body.

(iv) The increment in nominal stress normal to the middle surface is negligible in comparison with the increments in the inplane stresses.

(v) The effect of the transverse shear strains and rotary inertia are negligible.

(vi) The plate material is homogeneous, isotropic, incompressible and obeys Von Mises yield criterion.

(vii) The elastic deformations are assumed to follow Hook's law and for plastic behaviour, the approximation made by Witmer *et al.* [9] that is the plastic stress–strain relationship is adequately represented by a linear strain hardening curve is employed.

(viii) Bauschinger effect is ignored.

(ix) The effect of strain rate on the material properties is neglected.

(x) The effects due to constraints at the edges are not considered. Therefore, ideal boundary conditions are assumed.

5. FORMULATION OF THE PROBLEM

It is convenient to think of the plate as being flat and free from stresses and strains. Let $0(x_i)$ be a right-handed frame of reference, the origin 0 being taken at the centre of the middle surface containing the $x_1 0 x_2$ plane and x_3 axis as vertical and pointing downwards. Wherever convenient, x_1, x_2, x_3 will be replaced by cylindrical polar co-ordinates r, θ, z respectively. For the latter triad of axes, the only non-zero metric tensors and christoffel symbols are [14]:

$$g^{11} = g_{11} = 1, \qquad g_{22} = \frac{1}{g^{22}} = r^2, \qquad g_{33} = g^{33} = 1 \tag{5.1}$$

and

$$\Gamma^1_{22} = -r, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}. \tag{5.2}$$

In view of the assumptions, the components of the unit normal to the local yield surface are taken as

$$m_{ij} = \begin{bmatrix} m & 0 & 0 \\ 0 & k_1 m & 0 \\ 0 & 0 & k_2 m \end{bmatrix} \tag{5.3a}$$

where

$$m^2(1+k_1^2+k_2^2) = 1 \tag{5.3b}$$

and

$$1+k_1+k_2 = 0 \tag{5.3c}$$

The last equation implies plastic incompressibility. Using these equations along with (5.1), (5.2), (3.6) and (3.5), the physical components of incremental true stresses are given by:

$$\mathrm{d}\sigma_{rr} = 2\mu(a_{11}\,\mathrm{d}e_{rr}+a_{12}\,\mathrm{d}e_{\theta\theta}+\mathrm{a}_{13}\,\mathrm{d}e_{zz})+\mathrm{d}p-2\sigma_{rz}\,\mathrm{d}\omega_{yz} \tag{5.4a}$$

$$\mathrm{d}\sigma_{\theta\theta} = 2\mu(a_{12}\,\mathrm{d}e_{rr}+a_{22}\,\mathrm{d}e_{\theta\theta}+a_{23}\,\mathrm{d}e_{zz})+\mathrm{d}p \tag{5.4b}$$

$$\mathrm{d}\sigma_{zz} = 2\mu(a_{13}\,\mathrm{d}e_{rr}+a_{23}\,\mathrm{d}e_{\theta\theta}+a_{33}\,\mathrm{d}e_{zz})+\mathrm{d}p-2\sigma_{rz}\,\mathrm{d}\omega_{zr} \tag{5.4c}$$

$$\mathrm{d}\sigma_{rz} = 2\mu\left(\mathrm{d}e_{rz}-\frac{\sigma_{rr}}{2\mu}\,\mathrm{d}\omega_{rz}-\frac{\sigma_{zz}}{2\mu}\,\mathrm{d}\omega_{zy}\right) \tag{5.4d}$$

where

$$\begin{aligned} a_{11} &= 1-\frac{m^2\delta}{(1+\delta)}; & a_{22} &= 1-m^2k_1^2\frac{\delta}{(1+\delta)} \\ a_{33} &= 1-m^2k_2^2\frac{\delta}{(1+\delta)}; & a_{12} &= -m^2k_1\frac{\delta}{(1+\delta)} \\ a_{13} &= -m^2k_2\frac{\delta}{(1+\delta)}; & a_{23} &= -m^2k_1k_2\frac{\delta}{(1+\delta)} \\ \delta &= \frac{2\mu}{H}; & \mathrm{d}\omega_{rz} &= -\mathrm{d}\omega_{zr} = \frac{1}{2}\left(\frac{\partial(\mathrm{d}u)}{\partial z}-\frac{\partial(\mathrm{d}w)}{\partial r}\right) \end{aligned} \tag{5.5}$$

and dp is the increment in hydrostatic pressure. Similarly, combining equations (2.4) with (5.1) and (5.2), the physical components of strain increments are:

$$de_{rr} = \frac{\partial(du)}{\partial r} \tag{5.6a}$$

$$de_{\theta\theta} = \frac{du}{r} \tag{5.6b}$$

$$de_{zz} = \frac{\partial(dw)}{\partial z} \tag{5.6c}$$

$$de_{rz} = \frac{1}{2}\left(\frac{\partial(dw)}{\partial r}+\frac{\partial(du)}{\partial z}\right). \tag{5.6d}$$

The equations of motion (2.12) for an incremental deformation from the initial flat configuration of the plate take the form:

$$\frac{\partial(d\sigma_{rr})}{\partial r}+\frac{\partial(d\sigma_{rz})}{\partial z}+\frac{d\sigma_{rr}-d\sigma_{\theta\theta}}{r} = 0 \tag{5.7a}$$

$$\frac{\partial(d\sigma_{rz})}{\partial r}+\frac{d\sigma_{rz}}{r} = \rho\frac{\partial^2(dw)}{\partial t^2}. \tag{5.7b}$$

Since the deformations considered are infinitesimally small, it is reasonable to assume that the plate would be in the elastic state after the first incremental deformation. Nondimensionalizing the co-ordinates by writing

$$\beta = \frac{r}{a} \quad \text{and} \quad \xi = \frac{z}{a} \tag{5.8}$$

and using equations (5.4)–(5.6), the equations (5.7a) and (5.7b) are reduced to

$$\frac{\partial}{\partial\beta}\frac{1}{\beta}\frac{\partial}{\partial\beta}(\beta\, du)+\frac{1}{2}\frac{\partial^2(du)}{\partial\xi^2} = \frac{1}{2}\frac{\partial^2(dw)}{\partial\xi\partial\beta} \tag{5.9a}$$

and

$$\frac{1}{\beta}\frac{\partial}{\partial\beta}\left(\beta\frac{\partial(du)}{\partial\xi}\right)+\frac{1}{\beta}\frac{\partial}{\partial\beta}\left(\beta\frac{\partial(dw)}{\partial\beta}\right) = \frac{\rho a^2}{\mu}\frac{\partial^2(dw)}{\partial t^2} \tag{5.9b}$$

where the increment in hydrostatic pressure has been evaluated from

$$ds^{33} = 0. \tag{5.10}$$

Elimination of du from equations (5.9a) and (5.9b) results in

$$\frac{\rho a^2}{\mu}L\frac{\partial^2(dw)}{\partial t^2}-L^2\, dw+\frac{\rho a^2}{2\mu}\frac{\partial^4(dw)}{\partial\xi^2\partial t^2}-L\frac{\partial^2(dw)}{\partial\xi^2} = 0 \tag{5.11}$$

where

$$L = \frac{1}{\beta}\frac{\partial}{\partial\beta}\beta\frac{\partial}{\partial\beta}.$$

The initial conditions for this problem can be expressed as

$$\frac{\partial(\mathrm{d}w)}{\partial t}\bigg]_{(\beta,\xi,0)} = V_0 \tag{5.12a}$$

$$\mathrm{d}w(\beta,\xi,0) = 0 \tag{5.12b}$$

$$\mathrm{d}u\,(\beta,\xi,0) = 0 \tag{5.12c}$$

and the boundary conditions during the incremental deformations are:

$$\mathrm{d}w\,(1,0,t) = 0 \tag{5.12d}$$

$$\mathrm{d}u\,(0,\xi,t) = 0 \tag{5.12e}$$

$$\frac{\partial(\mathrm{d}w)}{\partial\beta}\bigg]_{(0,\xi,t)} = 0 \tag{5.12f}$$

and

$$\mathrm{d}\sigma_{rr}(1,\xi,t) = 0 \tag{5.12g}$$

or

$$\mathrm{d}u\,(1,\xi,t) = 0 \tag{5.12h}$$

$$\frac{\partial(\mathrm{d}w)}{\partial\beta}\bigg]_{(1,\xi,t)} = 0 \tag{5.12k}$$

according as the edges are simply supported or rigidly clamped respectively. A solution of equation (5.11) which satisfies the requirements (5.12b)–(5.12f) is

$$\mathrm{d}w = \sin\left[\frac{2n}{a}\sqrt{\left(\frac{\mu}{3\rho}\right)}t\right][J_0(n\beta)\{B\cos n\xi - A\sin n\xi\} - BJ_0(n)] \tag{5.13a}$$

$$\mathrm{d}u = \sin\left[\frac{2n}{a}\sqrt{\left(\frac{\mu}{3\rho}\right)}t\right][J_1(n\beta)\{A\cos n\xi + \beta\sin n\xi\}] \tag{5.13b}$$

where J_m stands for Bessel function of order m. The two constants A and B are related because of the interaction between bending and stretching of the plate. The boundary conditions (5.12g) is satisfied if

$$3J_0(n) = J_2(n) \tag{5.14a}$$

and for the rigidly clamped edges, the conditions (5.12h) and (5.12k) are fulfilled if

$$J_1(n) = 0. \tag{5.14b}$$

Since both (5.14a) and (5.14b) have infinite roots, equations (5.13) would yield the displacement components for each root of these equations. However, motivated by the theoretical predictions [3] of the radial displacements for an impulsively loaded membrane and the favourable results obtained by Wierzbicki [8] by considering only the first mode of deformation, it is reasonable to assume that only the fundamental mode for which n has the value 2·2164 and 3·8317 respectively persists.

The solution of the equations of motion for the subsequent incremental deformations becomes increasingly more cumbersome because of the curvature of the deformed surfaces and the possibility of the onset of plastic flow in certain regions. Fortunately, the equations (5.13) can also be obtained by a simultaneous solution of

$$du^i{}_{,i} = 0 \tag{5.14c}$$

$$de_{13} = 0 \tag{5.14d}$$

under the same boundary conditions. Owing to the natural tendency of the particles to follow a continuous path, it is conceivable that during the incremental deformation, the motion of the particles would satisfy equations (5.14) with respect to the axes embedded in the material and deformed by the whole preceding strain. Therefore, the following displacement functions would also represent the incremental displacement components with respect to the deformed axes.

$$d\bar{w} = J_0(n\beta)[B\cos n\xi - A\sin n\xi] - BJ_0(n) \tag{5.15a}$$

$$d\bar{u} = J_1(n\beta)[A\cos n\xi + B\sin n\xi] \tag{5.15b}$$

To superimpose the incremental deformations, it is necessary to refer the local incremental strain and displacement components back to the fixed reference axes by employing the transformation laws (2.6) and (2.7). After the second stage of deformation, the expressions for the direct transformation of local components to the global axes become intricate and are difficult to handle. It is, therefore, advisable to successively refer the local components to the immediate previous stage of deformation. Some details of this method are given in Ref. [17].

6. BENDING AND STRETCHING

The deflection of the plate is invariably accompanied by the stretching of the middle surface, the exception being the case when the surface is developable. Considering the geometry of the deformed surface, the increase in its radius Δl is given by:

$$\Delta l = \int_0^a \left[\left\{1 + \left(\frac{\partial(dw)}{\partial r}\right)^2_{\xi=0}\right\}^{\frac{1}{2}} - 1\right] dr. \tag{6.1}$$

Alternatively, it may be expressed as

$$\Delta l = \int_0^a (d\underline{e}_{rr})_{\xi=0}\, dr \tag{6.2}$$

where $d\underline{e}_{rr}$ is the increment in natural strain at any radius of the middle surface. Using equations (5.15a) and (5.15b), the expression (6.1) and (6.2) when combined together give

$$\int_0^1 \left[\left\{1 + \left(\frac{Bn}{a}J_1(n\beta)\right)^2\right\}^{\frac{1}{2}} - 1\right] d\beta = \int_0^1 \ln\left(1 + \frac{AJ_1(n\beta)}{a\beta}\right) d\beta. \tag{6.3}$$

For small deformations, that is, if $(\partial(dw)/\partial r)_{\xi=0} \ll 1$, $(du/r)_{\xi=0} \ll 1$, relation (6.3) simplifies to

$$\frac{B^2n^2}{2a}\int_0^1 [J_1(n\beta)]^2\, d\beta = A\int_0^1 \frac{J_1(n\beta)}{\beta}\, d\beta. \tag{6.4}$$

It shows that the deformations of the middle surface are proportional to the square of the central deflection.

7. YIELD CRITERION

Plastic flow is assumed to commence at a point if the local stress distribution satisfies the condition

$$\bar{\sigma} = \sigma_0 \tag{7.1}$$

where σ_0 is the yield stress in a simple tension and $\bar{\sigma}$, the generalised stress is given by

$$\bar{\sigma}^2 = \tfrac{1}{2}[(\sigma_{rr}-\sigma_{\theta\theta})^2+(\sigma_{\theta\theta}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{rr})^2]. \tag{7.2}$$

The direction ratios (5.3) of the local normal to the yield surface can, therefore, be expressed as

$$k_1 = (2\sigma_{\theta\theta}-\sigma_{rr}-\sigma_{zz})/(2\sigma_{rr}-\sigma_{\theta\theta}-\sigma_{zz}) \tag{7.3}$$

$$k_2 = (2\sigma_{zz}-\sigma_{rr}-\sigma_{\theta\theta})/(2\sigma_{rr}-\sigma_{\theta\theta}-\sigma_{zz}). \tag{7.4}$$

For workhardening solids, it is assumed that the yield surface maintains its shape, centre and orientation but expands uniformly about the origin. Hence the direction ratios of the local normal to the yield surface would remain unaltered during the loading. Also during reversed loading, tensile yielding is assumed to start when

$$\bar{\sigma} = \bar{\sigma}^* \tag{7.5}$$

where $\bar{\sigma}^*$ (greater than the lower static yield stress in simple tension) is the generalized stress in compression when the unloading begins.

8. COMPUTATION AND DISCUSSION OF RESULTS

In order to verify the validity of the assumptions incorporated in the development of the theory, it is imperative to compute and compare results with the experimental findings or the predictions of the other theories. The first step in the numerical computation of results involves a proper choice of the increment in deformation. This is guided by the approximations made in simplifying the expression (2.1) for the strain increments and that involved in reducing (6.3) to (6.4). To safeguard against any error that might creep in because of these simplifications and also to minimise the computation time, the increment in deformation is gradually reduced till the difference in the two consecutive results is insignificant. An increment in deformation resulting in an increment in the central reflection of the middle surface of 0·2 times the thickness was used. Thus the constant B in equations (5.15) is determined from (5.15a). The other constant A occurring in these expressions is then obtained from the relation (6.4) in which both the integrations can be evaluated numerically. The resulting increments in displacement and strain are calculated using equations (5.15) and (5.6) respectively. The corresponding values with respect to the global axes are then determined by employing the transformation laws (2.6)–(2.8). The increase in the stresses at a point are evaluated with the help of the expressions (5.4). If the stress-state at a point satisfies the yield criterion (7.2), the values of k_1 and k_2 as determined from (7.3) and (7.4) are used. The stress–strain curve for the material of the plate fixes the value of δ. If the stress-state at a point is in the elastic region, δ is taken as zero. The changes in the values of the nominal stresses, required for the purpose of estimating the strain energy of deformation, are calculated from (2.10) and (2.11) in which everything else is known. For this purpose, the plate is considered to be made of a large number of small strips. The stress distribution throughout the strip is taken as uniform and equal to that at its centre. The number of divisions is adjusted to attain the desired accuracy. In the results stated here, the plate was divided into 100 equal parts along the radius and 10 equal parts along the thickness. For perfectly plastic solids, the stress-state at a point is taken to be frozen once the plastic flow starts. However, because of the local rotations of the particles during the deformation, the increment in nominal stresses would have non-zero though very small values. Thus the strain energy of deformation can be determined as a function of the central deflection. From equation (2.14), it follows that the central deflection can be related to the kinetic energy supplied to the plate.

The co-relation between the theoretical predictions and the corresponding experimental findings is restricted because of the assumptions of ideal edge conditions and the simultaneous axisymmetric loading of the entire plate. Both these are difficult to be accomplished experimentally. Also during the deformation of the plate, the strain rate is varying from point to point and also with time. Its effects, if any, on the material properties has not been considered in the analysis. These considerations, the use of kinematically admissible displacement field, neglect of shear deformations and of rotary inertia put some restrictions on the applicability of the theory. However, the analysis is flexible in the use of the yield criterion and also provides a solution to both the problems having different end conditions.

Using this approach, the ultimate deflection profile and the central deflection for varying amount of input energy are computed for the case of simply supported plates tested by Florence [18]. The plates used in these experiments can be characterised by the following mechanical and geometrical parameters.

TABLE 1

Material	σ_0 (lb/in.^{2})	ρ (lb-sec^2/in.^{4})	Thickness $2h$ (in.)	Radius a (in.)
C.R. steel 1018	79,000	$7{\cdot}32\times10^{-4}$	0·241	4·0

The theoretical predictions taking perfectly plastic material response are presented in Figs. 1 and 2. The computed values compare favourably with those observed experimentally. For the case of rigidly clamped plates, the data† from experiments dealing with underwater explosions is used for comparison purposes in Fig. 3. The details of these experiments may be found in [15]. Whereas the correlation between the computed and observed profiles is quite good near the centre, the two differ close to the edges. This discrepancy can possibly be due to the assumption of ideal edge conditions. Since experimental evidence for the kinetic energy input vs. central deflection for rigidly clamped plates is scarce, the results are compared with the bending solution of Wang and Hopkins [4] for perfectly plastic material behaviour ($E = 30\times10^6$ psi, $\rho = 7{\cdot}52\times10^{-4}$ lb sec^2/in.$^{-4}$, $a = 10$ in., $2h = \frac{1}{4}$ in. and $\sigma_0 = 60\times10^3$ psi, KE/volume imparted to plate $= \frac{1}{2}\rho V_0^2$). Because of less amount of energy needed for initial elastic deformations, their considerations should give deflections more than those predicted by rigid-plastic theory. Since the membrane energy

†$a = 10$ in., $2h = \frac{1}{4}$ in., $\sigma_0 = 49{,}000$ psi, mild steel.

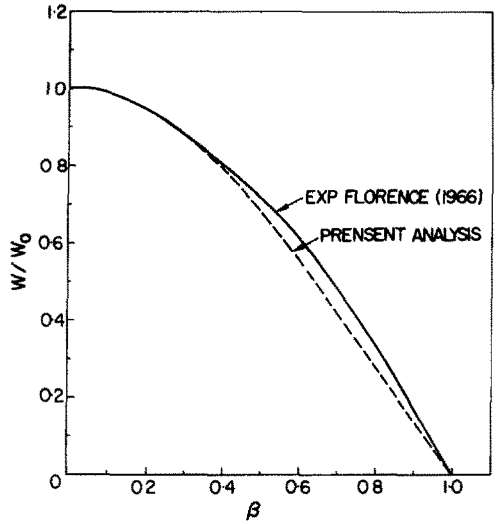

Fig. 1. Final deflection profile (simply supported plate).

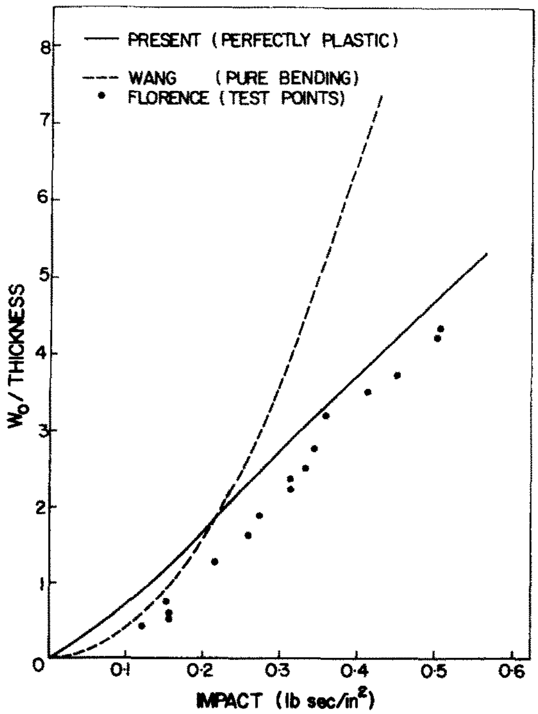

Fig. 2. Central deflection as a function of initial impact (simply supported).

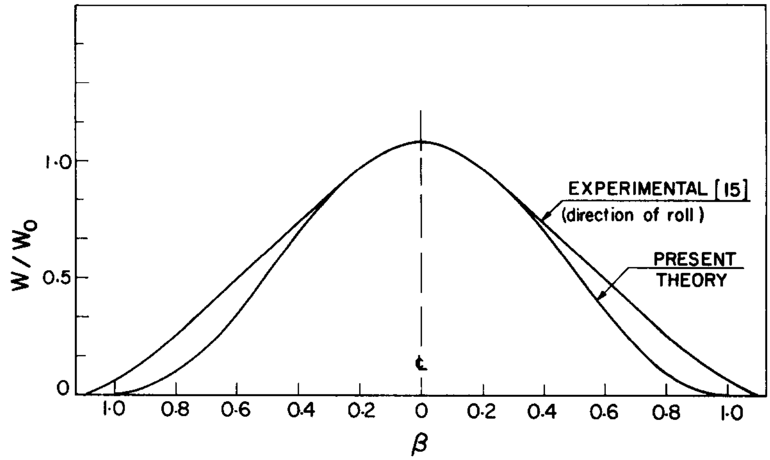

FIG. 3. Final deflection profile (rigidly clamped).

increases faster than that due to bending, the successive incremental deformations would require increasingly more energy. Therefore, a situation can be envisaged (comparable to point E in Fig. 4) when the actual energy of deformation is equal to that given by the pure bending theory based on rigid plastic behaviour. Below this stage, the latter analysis would underestimate deflections whereas greater values would be predicted for true deformations exceeding this particular situation. These considerations support the qualitative agreement between the results of present work and of Ref. [4].

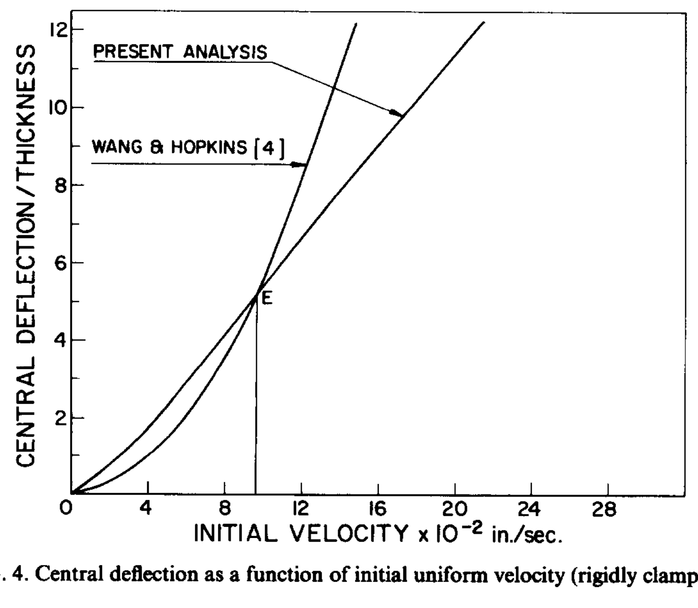

FIG. 4. Central deflection as a function of initial uniform velocity (rigidly clamped).

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REFERENCES

[1] G. E. HUDSON, A theory of the dynamic plastic deformation of a thin diaphragm. *J. appl. Phys.* **22**, 1 (1951).
[2] D. FREDRICK, A Simplified Analysis of Circular Membranes Subjected to an Impulsive Loading Producing Large Plastic Deformations, *Proc. 4th Annual Conf. Solid Mech.*, University of Texas, Austin, Texas (1959).
[3] D. E. BOYD, Dynamic deformations of circular membranes. *J. Engng Mech. Div. Am. Soc. civ. Engrs* **92**, 1 (1966).
[4] A. J. WANG and H. G. HOPKINS, On the plastic deformation of built in circular plates under impulsive load. *J. Mech. Phys. Solids* **3**, 22 (1954).
[5] A. J. WANG, The permanent deflection of a plastic plate under blast loading. *J. appl. Mech.* **22**, 365 (1955).
[6] H. G. HOPKINS and W. PRAGER, On the dynamics of plastic circular plates. *J. appl. Math. Phys.* **5**, 317 (1954).
[7] N. JONES, Impulsive loading of a simply supported circular rigid plastic plate. *J. appl. Mech.* **27**, 59 (1968).
[8] T. WIERZBICKI, Large deflection of a strain rate sensitive plate loaded impulsively. *Archum Mech. stosow.* **21**, 67 (1969).
[9] E. A. WITMER *et al.*, Large dynamic deformations of beams, circular rings, circular plates and shells. *AIAA Jnl* **1**, 1848 (1963).
[10] L. BRIOULLIN, *Tensors in Mechanics and Elasticity*, p. 294. Academic Press (1964).
[11] R. HILL, A general theory of uniqueness and stability in elastic plastic solids. *J. Mech. Phys. Solids* **6**, 236 (1958).
[12] R. HILL, Some basic principles in the mechanics of solids without a natural time. *J. Mech. Phys. Solids* **7**, 209 (1959).
[13] J. G. OLDROYD, Non-Newtonian effects in elastico–viscous liquids. *Proc. R. Soc.* **A245**, 278 (1958).
[14] A. E. GREEN and W. ZERNA, *Theoretical Elasticity.* Clarendon Press (1954).
[15] J. A. LISANBY and J. E. RASMUSSEN, The Relative Strengthening of Steels at the High Strain Rates of Explosive Loading, M.Sc. Thesis, M.I.T. (1956).
[16] W. PRAGER, Theory of plastic flow versus theory of plastic deformation. *J. appl. Phys.* **19**, 540 (1948).
[17] R. C. BATRA, Elastoplastic Deformations of Impulsively Loaded Circular Plates, M.A.Sc. Thesis, p. 44, University of Waterloo, Waterloo, Ontario, Canada (1969).
[18] A. L. FLORENCE, Circular platc under a uniformly distributed impulse. *Int. J. Solids Struct.* **2**, 37 (1966).

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Абстракт—Исследуется динамическое поведение упруго-пластических круглых пластинок, с прогибами в таких то областях, где как изгибные моменты так и мембранные усилия являются значительными. Формулировка заключается к двухмерным и осесимметрическим поведениям. Не учитываются деформации сдвига, инерция вращения и чувствительность к скорости деформации материала. Решаются уравнения движения для малых деформаций, исходя из начальной плоской конфигурации. Рассматривается влияние кривизн деформированной поверхности на последующие деформации, предполагая, что последовательные деформации случаются относительно конвективных осей. Затем, используется кинематическое Допускаемое поле скоростей, согласное с вышеупомянутым решением, для описания поведения пластинки при больших деформациях. Доводится суперпозиция последовательных приращений для перемещений и деформаций, путем отнесения каждого перемещения и деформации к заданным общим осям. Пользуясь этой техникой, дается расчет деформированной формы пластинки и начальной скорости, представленных функцией центрального прогиба. Далее, эти величины сравниваются с соответствующими экспериментальными результатами.