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MORPHOLOGICAL STABILITY OF A PROPAGATING DOMAIN WALL IN TWO-DIMENSIONAL FERROELASTIC TRANSFORMATIONS

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Abstract—We have studied the morphological stability of a propagating domain wall separating two uniformly deformed regions of a ferroelastic material undergoing plane strain deformations. Following the literature, the strain energy density for the ferroelastic material is assumed to be a sum of three functions: one quadratic in the shear strain, the second quadratic in the dilatational strain and the third a Landau-type nonlinear function of deviatoric normal strains. The general equations for linearized morphological stability of a planar propagating interface are derived. It is shown that the propagating planar domain wall is stable against infinitely long-wave perturbations. To examine the relationship between the morphological stability and the propagating speed, we consider a special class of ferroelastic materials and show that there is a critical value m^* such that the steadily propagating domain wall is stable for $m < m^*$, where $m = \rho V_0^2/\mu$; ρ, μ and V_0 equal respectively the mass density, shear modulus and the propagation speed of the interface.

1. INTRODUCTION

A ferroelastic material (Aizu, 1969) is one that exhibits, in the absence of external mechanical loads, two or more stable states characterized by different spontaneous strains, and the material in one stable state can be transformed into the other stable state by applying mechanical loads. Phase transformations in such materials have been studied by using the Ginzburg–Landau theory, wherein the material response was assumed to be nonlinear in the deviatoric part of the deformation but linear in the dilatational and shear deformations [e.g. see Barsch and Krumhansl (1988) and Jacobs (1985, 1992)]. They found soliton-type domain walls corresponding to the first-order dilationless and shearless phase transformations. When all coefficients of the strain gradient terms vanish, i.e. when the Ginzburg–Landau model reduces to the Landau model, these domain walls reduce to the planar surfaces separating two stable uniformly deformed regions. This is somewhat akin to the directional solidification process studied by Mullins and Sekerka (1964), Langer (1980) and Godreche (1993), wherein a moving planar interface separates the solid and liquid regions. It can become morphologically unstable and then develop into a cellular or dendritic pattern. Therefore, to delineate the physical admissibility of the solutions obtained by Barsch and Krumhansl (1988) and Jacobs (1985, 1992), their morphological stabilities should be examined. To our knowledge this has not been studied. We note that a planar interface is said to be morphologically stable if infinitesimal geometric perturbations of its planar form die out in time.

The study of the stability of phase transformations is generally difficult. As one of few examples, Magrari (1983) has studied the "energetic stability" of the one-dimensional soliton-type solution for the Ginzburg–Landau model of shape memory alloys. Here, we employ the Landau model and use Mullins and Sekerka's method [e.g. see Mullins and Sekerka (1964), Godreche (1993) and Langer (1980)] to study the morphological stability of a propagating interface or a domain wall separating two stable uniform phases of these ferroelastic materials, with emphasis on the relationship between the stability and the propagation speed.

The propagation of the interphase boundary has been studied for one-dimensional shape memory alloys (Falk and Seibel, 1987), for one-dimensional solids (James, 1980; Hutchinson and Neale, 1983; Coleman, 1985), and for two-dimensional elasto-plastic materials (Fager and Bassani, 1986; Tugcu and Neale, 1987). For materials undergoing martensitic transformations, Nishiyama (1978) has pointed out that the morphology of the interface between two phases is related to its propagation speed. Here, we present a quantitative analysis of the morphological stability of propagating domain walls in ferroelastic transformations discussed in Barsch and Krumhansl (1988) and Jacobs (1985, 1992).

The general equations for linearized morphological stability of a planar propagating interface are derived. For the sake of simplicity, we focus on a special class of ferroelastic materials, for which it is found that there is a critical value m^* of $m = \rho V_0^2/\mu$, where ρ is the mass density, V_0 the propagating speed and μ the shear modulus, such that the domain wall is morphologically stable for $m < m^*$ and is unstable for $m > m^*$. Thus, this example shows that the morphological stability of the propagating domain wall in ferroelastic transformations is determined by its propagation speed.

2. BASIC EQUATIONS

We use rectangular Cartesian coordinates $\mathbf{x}$ to study two-dimensional, infinitesimal, plane strain deformations of an elastic body and, therefore, neglect the effect of the change of configuration during phase transition. Three non-zero components ε_{ij} of the infinitesimal strain tensor $\underset{\sim}{\varepsilon}$ are related to the displacement $\underset{\sim}{u}$ by

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2, \quad i,j \equiv 1,2, \tag{1}$$

where a comma followed by an index j indicates partial differentiation with respect to x_j. Following Barsch and Krumhansl (1988) and Jacobs (1985, 1992), we assume that the strain energy density W can be expressed as

$$W = F(\eta_2) + A\eta_1^2 + B\eta_3^2, \tag{2}$$

where

$$\eta_1 = \varepsilon_{11} + \varepsilon_{22}, \quad \eta_2 = \varepsilon_{11} - \varepsilon_{22}, \quad \eta_3 = \varepsilon_{12}, \tag{3}$$

A and B are positive material constants, and $F(\cdot)$ is a Landau-type nonlinear function of η_2 such that its derivative,

$$f(\eta_2) = \frac{\mathrm{d}F}{\mathrm{d}\eta_2}, \tag{4}$$

is a "rising–falling–rising" function of η_2, as shown in Fig. 1. We assume that $f(\eta_2) = 0$ has at least two roots located on the rising (stable) parts of the curve. We note that Barsch and Krumhansl (1988) and Jacobs (1985, 1992) included strain gradient terms in the expression for the strain energy density to describe the structure of the interface, and employed it to study phase transformations in Nb3Sn, V3Si and In–Tl alloys.

From the expression (2) for the strain energy density, we obtain the following for the components σ_{ij} of the stress tensor $\underset{\sim}{\sigma}$:

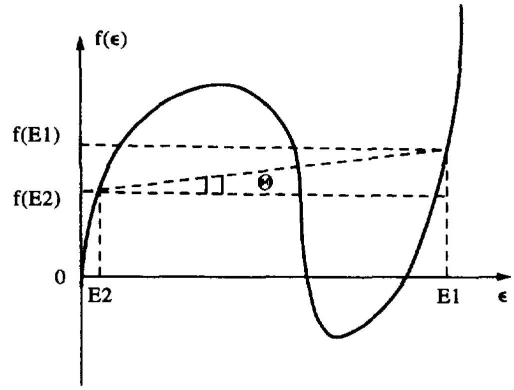

Fig. 1. Constitutive function $f(*)$.

$$\sigma_{11} = \frac{\partial W}{\partial \varepsilon_{11}} = f(\varepsilon_{11} - \varepsilon_{22}) + 2(\varepsilon_{11} + \varepsilon_{22})A \tag{5a}$$

$$\sigma_{22} = \frac{\partial W}{\partial \varepsilon_{22}} = -f(\varepsilon_{11} - \varepsilon_{22}) + 2(\varepsilon_{11} + \varepsilon_{22})A \tag{5b}$$

$$\sigma_{12} = \frac{1}{2}\frac{\partial W}{\partial \varepsilon_{12}} = B\varepsilon_{12}. \tag{5c}$$

With eqns (5), the equations expressing the balance of linear momentum become

$$\rho \ddot{u}_1 = f'(\eta_2)(u_{1,11} - u_{2,12}) + 2A(u_{1,11} + u_{2,12}) + B(u_{1,22} + u_{2,12})/2 \tag{6a}$$

$$\rho \ddot{u}_2 = -f'(\eta_2)(u_{1,12} - u_{2,22}) + 2A(u_{1,12} + u_{2,22}) + B(u_{1,12} + u_{2,11})/2, \tag{6b}$$

where ρ is the constant mass density and a superimposed dot indicates partial differentiation with respect to time t.

3. STEADY SHOCK WAVES

Ferroelastic materials described by eqn (2) can exhibit a shock wave-like deviatoric transformation for which a propagating planar interface (domain wall) separates two uniformly deformed regions.

Barsch and Krumhansl (1988) and Jacobs (1985, 1992) have provided motivations for studying dilatationless and shearless steady shock wave solutions for which η_2 is piecewise constant. We also study dilatationless and shearless steady shock waves. Thus,

$$\ddot{u}_i = 0, \quad i = 1, 2, \tag{7}$$

$$\eta_1 = \eta_3 = 0, \quad \text{and} \quad \eta_2 = \text{const.} \tag{8}$$

Let the normal to the interface which propagates at the constant speed V_0 make an angle

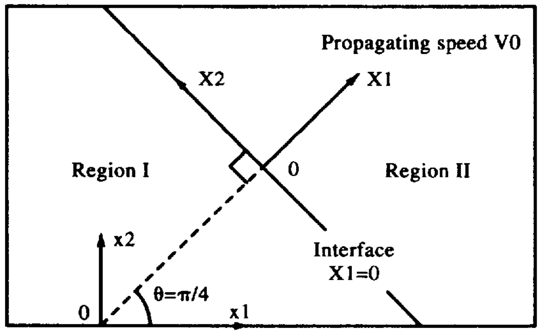

Fig. 2. Propagating domain wall.

θ with the x_1-axis, as shown in Fig. 2. In the fixed (x_1, x_2, t) frame, the equation of the moving interface is

$$x_2 - V_0 t \sin\theta = m(x_1 - V_0 t \cos\theta), \tag{9}$$

where

$$m = -\mathrm{ctg}\,\theta. \tag{10}$$

The shock wave divides the deforming region into two uniformly deforming regions. That is,

$$\eta_2 = E_1, \quad \eta_1 = 0, \quad \eta_3 = 0 \quad \text{in region I} \tag{11}$$

$$\eta_2 = E_2, \quad \eta_1 = 0, \quad \eta_3 = 0 \quad \text{in region II}, \tag{12}$$

where regions I and II are, respectively, behind and in front of the shock wave. The displacements in these regions are given by

$$\begin{aligned} u_1 &= E_1(x_1 - V_0 t\cos\theta)/2 + F_1(x_2 - V_0 t \sin\theta) + G_1 \\ u_2 &= -E_1(x_2 - V_0 t \sin\theta)/2 - F_1(x_1 - V_0 t\cos\theta) + G_2 \end{aligned} \tag{13}$$

in region I, and

$$\begin{aligned} u_1 &= E_2(x_1 - V_0 t\cos\theta)/2 + F_2(x_2 - V_0 t \sin\theta) + H_1 \\ u_2 &= -E_2(x_2 - V_0 t \sin\theta)/2 - F_2(x_1 - V_0 t\cos\theta) + H_2 \end{aligned} \tag{14}$$

in region II. Of the eight constants E_1, F_1, G_1, G_2, E_2, F_2, H_1 and H_2, E_1 and E_2 are determined by the jump conditions across the shock wave.

We assume that displacements are continuous across the shock wave, i.e.

$$[\![u_i]\!] = 0 \text{ at the interface}, \tag{15}$$

where $[\![f]\!] = f^+ - f^-$, f^+ and f^- being the values of f on the positive and negative sides of the interface. When $E_1 \neq E_2$, the jump condition (15) is satisfied if and only if

$$m^2 = 1, \quad \text{or} \quad \sin\theta = \pm 1/\sqrt{2}. \tag{16}$$

Thus, the positive normal to the steady shock wave must make an angle of 45° to the horizontal axis. Jacobs (1985, 1992) obtained this result by using a different argument. When eqn (16) holds, constants F_1 and F_2 are related by

$$2m(F_1 - F_2) = (E_2 - E_1). \tag{17}$$

Without loss of generality, we assume henceforth that m in eqn (10) equals -1 and introduce a new coordinate system X_1, X_2 attached to the propagating interface by

$$\begin{aligned} X_1 &= (x_1 + x_2)/\sqrt{2} - V_0 t \\ X_2 &= (x_2 - x_1)/\sqrt{2}. \end{aligned} \tag{18}$$

In the new frame (X_1, X_2, t) the shock wave appears stationary. The displacement components U_i with respect to X_i are related to u_i by

$$U_1 = (u_1 + u_2)/\sqrt{2}, \quad U_2 = (u_2 - u_1)/\sqrt{2}, \tag{19}$$

and η_1, η_2 and η_3 are given by

$$\eta_1 = (U_{1,1} + U_{2,2}), \quad \eta_2 = -(U_{1,2} + U_{2,1}), \quad \eta_3 = (U_{1,1} - U_{2,2})/2, \tag{20}$$

where a comma followed by an index j indicates partial differentiation with respect to X_j.

By requiring that U_1 and U_2 vanish at the interface $X_1 = 0$, we obtain the following from eqns (13) and (14):

$$\begin{aligned} U_1 &= 0, \quad U_2 = -E_1 X_1 \quad \text{for} \quad X_1 < 0 \\ U_1 &= 0, \quad U_2 = -E_2 X_1 \quad \text{for} \quad X_1 > 0. \end{aligned} \tag{21a,b}$$

Therefore, $X_1 = 0$ is a propagating planar interface separating two uniformly deformed regions. It may become morphologically unstable and then develop into a complicated pattern.

The main objective of this paper is to study its morphological stability. In order to do that we first derive jump conditions across the steadily propagating interface.

Writing the equations of motion (6a) and (6b) in the (X_1, X_2, t) frame and using the Rankine–Hugoniot conditions [e.g. see Falk and Seibel (1987) and Abeyaratne and Knowles (1991)], we arrive at the following:

$$-2\rho v [\![\dot{U}_1 - V_0 U_{1,1}]\!] = [\![(4A - B) U_{2,2} + (4A + B) U_{1,1}]\!] \tag{22}$$

$$-2\rho v [\![\dot{U}_2 - V_0 U_{2,1}]\!] = [\![(4A - B) U_{1,2} - 2f(\eta_2)]\!]. \tag{23}$$

Here, v is the local speed along the X_1-direction of the steady shock wave. The balance of total internal energy requires that

$$\begin{aligned} -v [\![W + \rho((\dot{U}_1 - V_0 U_{1,1})^2 + (\dot{U}_2 - V_0 U_{2,1})^2)/2]\!] &= \\ [\![(B\eta_3 + 2A\eta_1)(\dot{U}_1 - V_0 U_{1,1}) - f(\eta_2)(\dot{U}_2 - V_0 U_{2,1})]\!]. \end{aligned} \tag{24}$$

We note that the jump conditions (22)–(24) are exactly valid for the straight interface $X_1 = \text{const}$.

For the steady shock wave, eqns (21), (23) and (24) reduce to

$$f(E_1) - f(E_2) = \rho V_0^2 (E_1 - E_2) \tag{25}$$

$$F(E_1) - F(E_2) + \rho V_0^2 (E_1^2 - E_2^2)/2 = f(E_1)E_1 - f(E_2)E_2, \tag{26}$$

from which we obtain the so-called "equal area" condition:

$$F(E_1)-F(E_2) = [f(E_1)+f(E_2)](E_1-E_2)/2. \tag{27}$$

When $|\rho V_0^2| \ll 1$, eqns (25) and (26) imply that

$$f(E_1) = f(E_2) = \Sigma^0 \tag{28}$$

$$F(E_1)-F(E_2) = \Sigma^0(E_1-E_2), \tag{29}$$

where Σ^0 equals the externally applied load. Equation (29) is the classical "Maxwell Rule" [e.g. see Ericksen (1975)].

On the stress $f(\varepsilon)$–strain ε curve shown in Fig. 1, let the straight line joining states $(E_1, f(E_1))$ and $(E_2, f(E_2))$ subtend an angle Θ with the horizontal axis. Then,

$$\tan\Theta = \rho V_0^2$$

follows from eqn (25). From the figure, we conclude that

$$\rho V_0^2 < f'(E_1), \quad \rho V_0^2 < f'(E_2), \tag{30}$$

which impose an upper limit on the speed of propagation of the interface.

4. MORPHOLOGICAL STABILITY OF THE PROPAGATING INTERFACE

We now use the Mullins and Sekerka (1964) method to study the morphological stability of the propagating interface. Accordingly, we consider infinitesimal perturbation,

$$X_1 = X^*(X_2, t) = \delta \cos kX_2\, e^{\omega t}, \tag{31}$$

of the interface $X_1 = 0$ in the reference frame (X_1, X_2, t) where δ and k are constants with $|\delta| \ll 1$, and ω is the rate of change of the pertubation. Therefore, the perturbation will decay and the interface will be stable if $\mathrm{Re}(\omega) < 0$, and for $\mathrm{Re}(\omega) > 0$ the pertubation will grow and the interface will become unstable. The linearized version of equations of motion (6) are

$$2\rho(\ddot{U}_1 - 2V_0\dot{U}_{1,1} + V_0^2U_{1,11}) = 2f'(E_i)U_{1,22} + (B+4A)U_{1,11} + (2f'(E_i)+4A-B)U_{2,12} \tag{32a}$$

$$2\rho(\ddot{U}_2 - 2V_0\dot{U}_{2,1} + V_0^2U_{2,11}) = 2f'(E_i)U_{2,11} + (B+4A)U_{2,22} + (2f'(E_i)+4A-B)U_{1,12}, \tag{32b}$$

where

$$f'(E_i) = f'(E_1) \quad \text{for} \quad X_1 < X^*$$
$$f'(E_i) = f'(E_2) \quad \text{for} \quad X_1 > X^*. \tag{33}$$

Equations (32) are uncoupled if

$$2f'(E_i)+4A-B = 0. \tag{34}$$

However, such a condition may not generally be satisfied.

The steady state solution (21) for $X_1 < X^*$ should be replaced by

$$U_1(X_1, X_2, t) = \tilde{u}_1(X_1)\sin kX_2\, e^{\omega t}$$

$$U_2(X_1, X_2, t) = -E_1 X_1 + \tilde{u}_2(X_1)\cos kX_2\, e^{\omega t}, \tag{35}$$

where $\tilde{u}_1$ and $\tilde{u}_2$ are of order δ. In order to get forms of $\tilde{u}_1$ and $\tilde{u}_2$, we first examine their eigensolutions. Let

$$\tilde{u}_1(X_1) = A_\lambda e^{\lambda X_1}, \quad \tilde{u}_2(X_1) = B_\lambda e^{\lambda X_1}, X_1 < X^*, \tag{36}$$

where A_λ and B_λ are λ-dependent coefficients. Substituting from eqn (35) into eqns (32), and recalling eqns (33), we obtain

$$(2\rho(\omega^2 - 2\omega V_0\lambda + V_0^2\lambda^2) + 2f'(E_1)k^2 - (4A+B)\lambda^2)A_\lambda + (4A - B + 2f'(E_1))\lambda k B_\lambda = 0 \tag{37}$$

$$(2\rho(\omega^2 - 2\omega V_0\lambda + V_0^2\lambda^2) - 2f'(E_1)\lambda^2 + (4A+B)k^2)B_\lambda - (4A - B + 2f'(E_1))\lambda k A_\lambda = 0. \tag{38}$$

The requirement that eqns (37) and (38) have a nontrivial solution yields a fourth-order equation,

$$[4A - B + 2f'(E_1)]^2\lambda^2 k^2 + \{2\rho(\omega^2 - 2\omega V_0\lambda + V_0^2\lambda^2) + 2f'(E_1)k^2 - [4A+B]\lambda^2\}\{2\rho(\omega^2 - 2\omega V_0\lambda + V_0^2\lambda^2) - 2f'(E_1)\lambda^2 + [4A+B]k^2\} = 0,$$

for the determination of λ. Having found λ, we can use either eqn (37) or (38) to ascertain A_λ/B_λ. Thus, assuming that two eigenvalues p and r with $\text{Re}(p) > 0$, $\text{Re}(r) > 0$ are admissible, eqns (35) should be of the form

$$U_1 = (\alpha e^{pX_1} + \bar{\varphi} e^{rX_1})\sin kX_2\, e^{\omega t} \tag{39a}$$

$$U_2 = -E_1 X_1 + (\bar{\alpha} e^{pX_1} + \varphi e^{rX_1})\cos kX_2\, e^{\omega t} \quad \text{for} \quad X_1 < X^*. \tag{39b}$$

Similarly, in the region $X_1 > X^*$, the steady-state solution (21b) should be replaced by

$$U_1 = (\beta e^{-qX_1} + \psi e^{-sX_1})\sin kX_2\, e^{\omega t} \tag{40a}$$

$$U_2 = -E_2 X_1 + (\bar{\beta} e^{-qX_1} + \bar{\psi} e^{-sX_1})\cos kX_2\, e^{\omega t}, \tag{40b}$$

where q and s are determined by the following fourth-order algebraic eigen-equation:

$$[4A - B + 2f'(E_2)]^2\eta^2 k^2 + \{2\rho(\omega^2 + 2\omega V_0\eta + V_0^2\eta^2) + 2f'(E_2)k^2 - [4A+B]\eta^2\}\{2\rho(\omega^2 + 2\omega V_0\eta + V_0^2\eta^2) - 2f'(E_2)\eta^2 + [4A+B]k^2\} = 0.$$

Here, α, $\bar{\alpha}$, β, $\bar{\beta}$, φ, $\bar{\varphi}$, ψ, $\bar{\psi}$ and δ are unknown constants, and p, q, r and s are determined from two analogous fourth-order equations with the constraints

$$\text{Re}(p) > 0, \quad \text{Re}(q) > 0, \quad \text{Re}(r) > 0 \quad \text{and} \quad \text{Re}(s) > 0, \tag{41}$$

where we have assumed that the boundary conditions at infinity are not perturbed. Substitution from eqns (39) and (40) into eqns (32) yields

$$[2\rho(\omega^2-2\omega V_0 p+V_0^2 p^2)+2f'(E_1)k^2-(4A+B)p^2]\alpha = -(4A-B+2f'(E_1))pk\bar{\alpha}$$
$$[2\rho(\omega^2-2\omega V_0 r+V_0^2 r^2)-2f'(E_1)r^2+(4A+B)k^2]\varphi = (4A-B+2f'(E_1))rk\bar{\varphi}$$
$$[2\rho(\omega^2+2\omega V_0 q+V_0^2 q^2)+2f'(E_2)k^2-(4A+B)q^2]\beta = (4A-B+2f'(E_2))qk\bar{\beta}$$
$$[2\rho(\omega^2+2\omega V_0 s+V_0^2 s^2)-2f'(E_2)s^2+(4A+B)k^2]\psi = -(4A-B+2f'(E_2))sk\bar{\psi}. \quad (42)$$

The remaining five equations for α, β, φ, ψ and δ are derived from the five jump conditions at the interface.

For infinitesimal perturbations (31) of the interface and of the corresponding solution (39) and (40), we make the following observations.

(a) Let $\hat{\theta}$ be the infinitesimal angle that the unit local normal $\mathbf{n}$ to the interface $X_1 = X^*(X_2, t)$ makes with the positive X_1-direction. Then,

$$\hat{\theta} = \frac{\partial X^*}{\partial X_2}$$
$$\partial_{X_1} = \cos\hat{\theta}\,\partial_{\mathbf{n}} + \sin\hat{\theta}\,\partial_{\mathbf{e}}$$
$$\partial_{X_2} = \cos\hat{\theta}\,\partial_{\mathbf{e}} - \sin\hat{\theta}\,\partial_{\mathbf{n}}$$
$$\cos\hat{\theta} \simeq 1+O(\delta^2), \quad \sin\hat{\theta} = O(\delta); \quad (43)$$

$\mathbf{e}$ is a unit vector tangent to the interface.

Furthermore,

$$\dot{U}_1 \simeq O(\delta), \quad U_{1,1} \simeq O(\delta), \quad \eta_1 \simeq O(\delta), \quad \eta_3 \simeq O(\delta)$$
$$\sin\hat{\theta} \simeq \frac{\partial X^*}{\partial X_2} = -k\delta \sin kX_2\, e^{\omega t}. \quad (44)$$

(b) The local speed, v, which essentially equals the normal speed v_n because of eqns (43), is given by

$$v = V_0 + \dot{X}^*(X_2, t). \quad (45)$$

Substituting from eqns (39) and (40) into the jump conditions (15), (22)–(24) and keeping in mind eqns (43)–(45), we arrive at the following:

$$\alpha+\bar{\varphi} = \beta+\bar{\psi}$$
$$(E_1-E_2)\delta = \bar{\alpha}+\varphi-\bar{\beta}-\psi$$
$$(4A+B-2\rho V_0^2)(p\alpha+r\bar{\varphi}+q\beta+s\bar{\psi}) = (4A-B+2\rho V_0^2)k(\bar{\alpha}+\varphi-\bar{\beta}-\psi)$$
$$f'(E_1)[(p\bar{\alpha}+r\varphi)+k(\alpha+\bar{\varphi})]+f'(E_2)[(q\bar{\beta}+s\psi)-k(\beta+\bar{\psi})]$$
$$+2\omega\rho V_0(\bar{\alpha}+\varphi-\bar{\beta}-\psi)-\rho V_0^2(p\bar{\alpha}+r\varphi+q\bar{\beta}+s\psi) = 0$$
$$\omega\rho V_0(E_1+E_2)(\bar{\alpha}+\varphi-\bar{\beta}-\psi)-k(\alpha+\bar{\varphi})[f(E_1)-f(E_2)]-\rho V_0^2[E_1(p\bar{\alpha}+r\varphi)+E_2(q\bar{\beta}+s\psi)]$$
$$+E_1 f'(E_1)[p\bar{\alpha}+r\varphi+k(\alpha+\bar{\varphi})]+E_2 f'(E_2)[q\bar{\beta}+s\psi-k(\beta+\bar{\psi})] = 0. \quad (46a–e)$$

We note that $2\rho V_0^2 k(\bar{\alpha}+\varphi-\bar{\beta}-\psi)$ on the right-hand side of eqn (46c) is the only additional term that appears when the straight interface $X_1 = 0$ is replaced by an infinitesimally curved interface X^* (X_2, t). Substitution for $\bar{\alpha}$, $\bar{\beta}$, $\bar{\varphi}$ and $\bar{\psi}$ from eqns (42) into eqns (46) yields five equations for five unknowns α, β, φ, ψ and δ. From the condition for the existence of a nontrivial solution of these equations, we obtain an expression for ω which determines the stability of the straight interface.

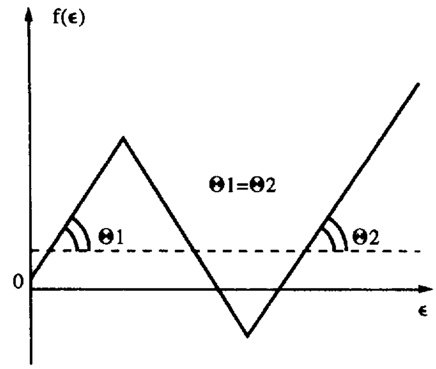

Fig. 3. A trilinear stress–strain relation with equal moduli for the "stable" parts.

As mentioned before, our interest is to examine the dependence of interface stability on the propagation speed. For simplicity, we study this problem with an example. As has been assumed by Aizu (1969), among others, the dependence of the modulus $f'(\varepsilon) \equiv \mathrm{d}f/\mathrm{d}\varepsilon$ upon the strain and also the change in its values during phase transformations are negligible. Therefore, we assume a trilinear material with two rising stable branches of equal slope:

$$f'(E_1) = f'(E_2) = \mu = \text{const.}, \tag{47}$$

as shown in Fig. 3, and consider a ferroelastic material for which

$$B+4A = 2f'(E) = 2\mu. \tag{48}$$

We note that it is not too severe a restriction upon the material parameters. For materials satisfying eqn (48), equations for λ and η reduce to

$$\begin{aligned} 2\rho(\omega^2-2\omega V_0\lambda+V_0^2\lambda^2)+2\mu k^2-(4A+B)\lambda^2 &= \pm i(4A-B+2\mu)\lambda k \\ 2\rho(\omega^2+2\omega V_0\eta+V_0^2\eta^2)+2\mu k^2-(4A+B)\eta^2 &= \pm i(4A-B+2\mu)\eta k, \end{aligned} \tag{49}$$

where p and s correspond to the "$-$" sign and r and q to the "$+$" sign on the right-hand side. The comparison of eqns (49) with eqns (42) yields

$$\bar{\alpha} = i\alpha, \quad \bar{\beta} = i\beta, \quad \bar{\varphi} = i\varphi, \quad \bar{\psi} = i\psi. \tag{50}$$

Define n by

$$n = (4A-B)/2\mu = (4A-B)/(4A+B), \tag{51}$$

where the second equality follows from eqn (48). Since A and B are positive numbers, therefore

$$1 \geqslant n \geqslant -1. \tag{52}$$

From eqns (46) we derive the following eigen-problem for α, φ, β and ψ:

$$\begin{aligned} \alpha+\bar{\varphi} &= \beta+\bar{\psi} \\ (p\bar{\alpha}+r\varphi-q\bar{\beta}-s\psi)+2k(\alpha+\bar{\varphi}) &= 0 \\ (2\omega\rho V_0+ik\rho V_0^2+n\mu ki)(p\bar{\alpha}+q\bar{\beta}) &= (2\omega\rho V_0-ik\rho V_0^2-in\mu k)(r\varphi+s\psi) \\ (2\omega\rho V_0-ik\rho V_0^2-in\mu k)(\bar{\alpha}-\bar{\beta})+(\mu-\rho V_0^2)(p\bar{\alpha}+q\bar{\beta}) &= 0. \end{aligned} \tag{53a–d}$$

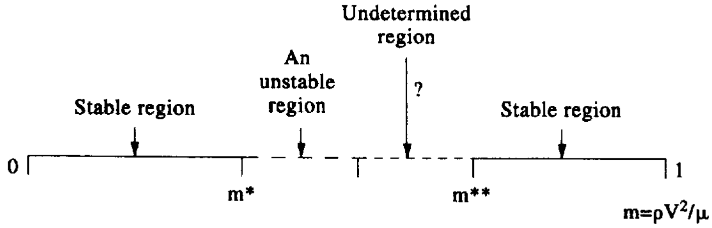

Fig. 4. Dependence of the domain of stability upon m.

Since eqn (53d) involves only $\bar{\alpha}$ and $\bar{\beta}$, to obtain the eigen-condition it is sufficient to derive another equation for $\bar{\alpha}$ and $\bar{\beta}$. This can be obtained from eqns (53) with the following result:

$$\bar{\alpha}[2\omega\rho V_0(2r(s+p)+2ik(p-r))-2ik(n\mu+\rho V_0^2)(p+r)(s-ik)] =$$
$$\bar{\beta}[2\omega\rho V_0(2s(r+q)+2ik(s-q))-2ik(n\mu+\rho V_0^2)(q+s)(r+ik)]. \quad (54)$$

The necessary and sufficient condition for eqns (53) and (54) to have a nontrivial solution is that

$$2\omega\rho V_0(rp-sq+ik(p-r+q-s))+ik(n\mu+\rho V_0^2)(qr-ps+ik(p+q+r+s)) =$$
$$(\mu-\rho V_0^2)(pq(r+s)+rs(p+q)+ik(ps-rq)), \quad (55)$$

where the constants p, q, r and s depend upon ω through eqns (49). From eqns (49) we obtain

$$\begin{aligned} 2(\rho V_0^2-\mu)r &= 2\omega\rho V_0+i(1+n)\mu k-\sqrt{\Delta_1} \\ 2(\rho V_0^2-\mu)s &= -(2\omega\rho V_0+i(1+n)\mu k)-\sqrt{\Delta_1} \\ 2(\rho V_0^2-\mu)p &= 2\omega\rho V_0-i(1+n)\mu k-\sqrt{\Delta_2} \\ 2(\rho V_0^2-\mu)q &= -(2\omega\rho V_0-i(1+n)\mu k)-\sqrt{\Delta_2}, \end{aligned} \quad (56)$$

where

$$\begin{aligned} \Delta_1 &= (2\omega\rho V_0+i(1+n)k\mu)^2-4(\rho V_0^2-\mu)(\rho\omega^2+\mu k^2) \\ \Delta_2 &= (2\omega\rho V_0-i(1+n)k\mu)^2-4(\rho V_0^2-\mu)(\rho\omega^2+\mu k^2), \end{aligned} \quad (57)$$

and in taking the square-root of Δ_1 and Δ_2 in eqns (56), roots with the positive real parts are kept.

It is important to note that the necessary and sufficient conditions for the existence and uniqueness of four roots p, q, r and s of eqns (56) are that

$$|\mathrm{Re}(\Delta_1)| > |\mathrm{Re}(2\omega\rho V_0)|, \quad |\mathrm{Re}(\Delta_2)| > |\mathrm{Re}(2\omega\rho V_0)|. \quad (58)$$

From eqns (55)–(57), we can derive

$$(\sqrt{\Delta_2}+\sqrt{\Delta_1})[2(\rho V_0^2-\mu)(\rho\omega^2+k^2(\mu-\rho V_0^2-n\mu))+\sqrt{\Delta_2}\sqrt{\Delta_1}] =$$
$$4i\omega\rho V_0 k(\rho V_0^2-\mu)(\sqrt{\Delta_1}-\sqrt{\Delta_2}). \quad (59)$$

Thus, if ω satisfies eqn (59), then $(-\omega)$ also satisfies it. Multiply eqn (59) by $(\sqrt{\Delta_1}-\sqrt{\Delta_2})$, cancel the common factor ω, and after some simplification we arrive at

$$(m+n)\sqrt{(\sqrt{2m\Omega}+i(1+n))^2-2(m-1)(\Omega+2)}\sqrt{(\sqrt{2m\Omega}-i(1+n))^2-2(m-1)(\Omega+2)}$$
$$= (1-m)(1-n)(2m-\Omega+n-1), \quad (60)$$

where recalling eqns (30) and (47),

$$\Omega = 2\rho\omega^2/\mu k^2, \quad m = \rho V_0^2/\mu, \quad 0 \leqslant m < 1 \quad (61a,b)$$

and the square-root is to be understood as the square-root of the positive real part. Squaring both sides of eqn (60), we get a quadratic equation in Ω whose solution is

$$\Omega = \frac{(1-n)^2(1-m)^2(2m+n-1)+2(n+m)^2[4(1-m)+(2m-1)(1+n)^2] \pm (n+m)(1+n)\sqrt{\Delta}}{(1-n)^2(1-m)^2-4(n+m)^2}, \quad (62)$$

where for $1 > m \geqslant 0$ and $1 \geqslant n \geqslant -1$,

$$\Delta = 16m(1-m)(n+m)^2[4-(1+n)^2]+8m(1-n)^2(1-m)^2(2m+n-1)$$
$$+(1-n)^4(1-m)^2 \geqslant 0. \quad (63)$$

Equations (62) and (63) imply that Ω is always a real number. We emphasize that the admissible root ω obtained from eqn (62) must satisfy the original eqn (59) and the restrictions (58). In particular, the nonexistence of such a root implies stability.

In order to study the stability of the interface, we first note that for $m = 0$, eqn (62) gives

$$\Omega = (n^2-1) \leqslant 0. \quad (64)$$

The inequality follows from (52). From eqn (61a), we conclude that ω is pure imaginary and, thus, the static interface is morphologically stable. For $m = 1$,

$$\Omega = -(1+n)^2/2 \leqslant 0,$$

and, again, the interface is morphologically stable. Loosely speaking, we may say that the interface is stable when m is sufficiently small or sufficiently close to 1.

Since Ω is a real number, the stability of the interface changes at the zeros of $\Omega = 0$. Recalling eqn (62), we find that there are only two admissible roots ($0 \leqslant m \leqslant 1$) m^* and m^{**} of $\Omega = 0$:

$$m^* = (\sqrt{2-n^2}-n)/2 \quad (65a)$$

$$m^{**} = (3-5n+(1+n)\sqrt{(2n^2-4n+3)})/(2(3-n)). \quad (65b)$$

Therefore, when m increases from zero, Ω becomes positive when $m^{**} > m > m^*$ and negative again when $m > m^{**}$. However, only m^* satisfies the original eqn (59). It can be shown that

$$\mathrm{d}m^*(n)/\mathrm{d}n \leqslant 0, \quad (66)$$

and m^* changes monotonically from 1 to 0 when n changes from -1 to 1.

The restrictions (58) are satisfied for $\Omega = 0$ if and only if $m < m^0$, where

$$m^0 = 1-(1+n)^2/4, \tag{67}$$

and one can show that $m^* < m^0 < m^{**}$ for $n^2 < 1$. Therefore, to find the critical speed, we confine ourselves to the admissible unstable case,

$$m^* < m < m^0. \tag{68}$$

Recalling that n defined by eqn (51) depends upon the values of material parameters A and B, we consider three values of n, namely, $n = 0$, $1/2$ and $-1/2$. For $n = 0$, we get

$$m^* = \frac{1}{\sqrt{2}} \simeq 0.71, \quad m^0 = 0.75, \quad m^{**} = (3+\sqrt{3})/6 \simeq 0.78. \tag{69}$$

For $n = 1/2$,

$$m^* \simeq 0.37, \quad m^0 \simeq 0.44 \quad \text{and} \quad m^{**} \simeq 0.47. \tag{70}$$

For $n = -1/2$,

$$m^* \simeq 0.86, \quad m^0 \simeq 0.94 \quad \text{and} \quad m^{**} \simeq 0.95. \tag{71}$$

For each of these three cases, the interface is found to be stable for $m < m^*$ and unstable for $m^* < m < m^0$.

Thus, we conclude that there is a nonvanishing range $(m^*, \bar{m})$ of values of m for which the interface becomes morphologically unstable; the value of $\bar{m}$ depends upon the material being studied; this is depicted schematically in Fig. 4. Therefore, the critical value of the propagation speed is given by m^*. The value of m^* approaches zero as $n \to 1$, which will be the case for $B \ll A$.

We now briefly discuss the case of infinite long-wave perturbations with $k = 0$ for the general case. It follows from eqns (32) that

$$\bar{\alpha} = \bar{\varphi} = \bar{\beta} = \bar{\psi} = 0 \tag{72}$$

$$\omega = -r(\pm\sqrt{f'(E_i)} - V_0\sqrt{\rho})/\sqrt{\rho}, \quad \omega = -s(V_0\sqrt{\rho} \pm \sqrt{f'(E_i)})/\sqrt{\rho} \tag{73}$$

and the interface becomes unstable when and only when the "$-$" sign is chosen in eqn (73). Using eqns (72) and (73), from eqns (46) one can derive the following:

$$f'(E_i) + \rho V_0^2 = 0,$$

which cannot hold, thereby implying that there is no solution of eqns (46). Hence, the interface is always stable against infinite long-wave perturbations. This conclusion is independent of condition (48).

5. CONCLUSIONS

We have used the Mullins and Sekerka method to analyse the morphological stability of a steady planar domain wall propagating in a ferroelastic material undergoing plane strain deformations. The material on the two sides of the domain wall is in different uniform states of deformation, which may be thought of as representing two different phases of the ferroelastic material. After having derived the general equations characterizing the stability of the propagating interface, specific results are obtained for a trilinear material for which the nonlinear part of the stress–strain curve is trilinear with two rising stable parts having the same slope, connected by an unstable part.

The straight interface is found to be morphologically stable against infinite long-wave perturbations. For a class of materials for which the material parameters satisfy the relation (48), our results show that there is one special value of m, namely m^* given by eqn (65a), such that the interface is stable when $m < m^*$ and becomes unstable as soon as m exceeds m^*. Therefore, m^* is the critical value for interface stability. Thus, the morphological stability of a propagating domain wall in a ferroelastic material depends on its propagation speed.

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