Journal of Sound and Vibration (2001) **241**(5), 913–919
doi:10.1006/jsvi.2000.3319, available online at http://www.idealibrary.com on IDEAL®

LETTERS TO THE EDITOR

EFFECT OF INERTIA FORCES ON THE DAMPING OF A CONSTRAINED LAYER FINITELY DEFORMED IN SHEARING

R. C. Batra and J.-H. Yu

Department of Engineering Science and Mechanics, MC 0219, *Virginia Polytechnic Institute and State University, Blacksburg, VA* 24061, *U.S.A.*

(*Received* 1 *June* 2000, *and in final form* 17 *July* 2000)

We analyze the effect of inertia forces on damping induced during large shearing deformations of an isotropic, incompressible and homogeneous viscoelastic layer constrained between two rigid circular cylinders. The layer is deformed by holding the inner solid cylinder fixed and rotating the outer one by applying to it a time-harmonic axisymmetric tangential velocity. It is assumed that the layer sticks to the cylinder walls and the length of each cylinder is very large as compared to the inner radius of the outer cylinder so that the end effects can be neglected. Thus, a material particle undergoes only tangential displacement which is assumed to be a function of the radial co-ordinate, r and time, t. That is,

$$r = R, \quad \theta = \Theta + f(r, t), \quad z = Z \tag{1}$$

represent the deformation field. The unknown function, f, is found by satisfying the balance of linear momentum,

$$\begin{aligned} \frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} &= \rho(\ddot{r} - r\dot{\theta}^2), \\ \frac{\partial T_{r\theta}}{\partial r} + \frac{2}{r} T_{r\theta} &= \rho(r\ddot{\theta} + 2\dot{r}\dot{\theta}), \end{aligned} \tag{2}$$

and the boundary conditions

$$u_\theta(r_1, t) = 0, \quad u_\theta(r_2, t) = r_2 \Lambda_0 \cos \omega t. \tag{3}$$

Here we have used a cylindrical co-ordinate system with the origin at the centre of the inner cylinder, and (r, θ) are the co-ordinates in the present configuration of the material particle that occupied the place (R, Θ) in the stress-free reference configuration. Furthermore, T_{rr}, $T_{\theta\theta}$ and $T_{r\theta}$ are the physical components of the Cauchy stress tensor, $\mathbf{T}$, and u_r and u_θ are the physical components of the displacement vector, $\mathbf{u}$. In equations (2) a superimposed dot indicates the material time derivative. The balance of mass is identically satisfied by the assumed displacement field. We presume that sufficient time has elapsed so that the applied time-harmonic tangential displacement $(3)_2$ produces a time-harmonic displacement field within the layer. Thus, initial conditions are not needed.

0022-460X/01/150913 + 07 $35.00/0

Equations (1)–(3) are supplemented by the following constitutive relation for the viscoelastic layer:

$$\mathbf{T} = -p(r,t)\mathbf{1} + \beta\mathbf{B} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{4}$$

Here p is the hydrostatic pressure not determined by the deformations of the body and is generally non-zero in the stress-free reference configuration, β equals the shear modulus in quasistatic deformations at zero strain, $\mathbf{B}$ is the left Cauchy–Green tensor, μ is the classical Newtonian viscosity, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first and second Rivlin–Ericksen tensors, and α_1 and α_2 are higher order viscosities. The constitutive relation (4) with $\mu = \alpha_1 = \alpha_2 = 0$ represents a neo-Hookean elastic solid, and with $\beta = 0$ a fluid of grade 2. Fosdick *et al.* [1] and Fosdick and Yu [2] have investigated shock waves, stability, and non-linear oscillations in bodies made of the viscoelastic material described by equation (4). Markovitz and Coleman [3] have analyzed the Couette flow of a second order fluid, which is different from the second grade fluid (see, e.g., Truesdell and Noll [4]). We note that the second law of thermodynamics expressed as the Clausius–Duhem inequality requires that $\mu \geqslant 0$, $\alpha_1 \geqslant 0$, and $\alpha_1 + \alpha_2 = 0$. Henceforth, we set $\alpha_1 = -\alpha_2$.

For the deformation field (1), the physical components in the (r, θ) plane of the deformation gradient, $\mathbf{F}$, the left Cauchy–Green tensor, $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, and the two Rivlin–Ericksen tensors, $\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^{\mathrm{T}} = 2\mathbf{D}$, and $\mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{A}_1$, where $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ have the following expressions:

$$\mathbf{F} = \begin{bmatrix} 1 & 0 \\ rf_{,r} & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & rf_{,r} \\ rf_{,r} & 1 + (rf_{,r})^2 \end{bmatrix},$$
$$2\mathbf{D} = \mathbf{A}_1 = \begin{bmatrix} 0 & rf_{,rt} \\ rf_{,rt} & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 2r^2(f_{,rt})^2 & rf_{,rtt} \\ rf_{,rtt} & 0 \end{bmatrix}, \quad \mathbf{A}_1^2 = \mathbf{0}. \tag{5}$$

Here, a comma followed by r denotes the partial derivative with respect to r.

We note that $D_{r\theta} = D_{\theta r}$ is the only non-zero component of the strain-rate tensor, $\mathbf{D}$. Thus, the energy dissipation rate per unit volume given by $tr(\mathbf{T}^v\mathbf{D})$ is determined by $T^v_{r\theta}$, where $\mathbf{T}^v$ equals the viscous part of the stress and is given by the sum of the last three terms on the right-hand side of equation (4). Hence, in order to compute the dissipation, we need not find the hydrostatic pressure, p. The deformation field, f, is determined by equation $(2)_2$ which reduces to

$$\left(\frac{\partial}{\partial r} + \frac{2}{r}\right)(\beta r f_{,r} + \mu r f_{,rt} + \alpha_1 r f_{,rtt}) = \rho r f_{,tt}. \tag{6}$$

Note that even when $\alpha_2 \neq -\alpha_1$ the material coefficient α_2 does not appear in the equation of motion (6) because for the deformation field (3) all terms in $\mathbf{A}_1^2$ vanish.

In terms of the non-dimensional variables

$$\bar{r} = \frac{r}{r_1}, \quad \bar{r}_2 = \frac{r_2}{r_1}, \quad \bar{\beta} = \frac{\beta}{\beta_0}, \quad \tau = \sqrt{\frac{\rho_0 r_1^2}{\beta_0}},$$
$$\bar{t} = \frac{t}{\tau}, \quad \bar{\omega} = \omega\tau, \quad \bar{\mu} = \frac{\mu}{\beta_0\tau}, \quad \bar{\alpha}_1 = \frac{\alpha_1}{\beta_0\tau^2}, \quad \bar{\rho} = \frac{\rho}{\rho_0},$$
$$\bar{\mathbf{T}} = \frac{\mathbf{T}}{\beta_0}, \quad \bar{\mathbf{D}} = \mathbf{D}\tau, \quad \bar{\mathbf{A}}_2 = \mathbf{A}_2\tau^2, \tag{7}$$

equation (6) becomes

$$\left(3 + r\frac{\partial}{\partial r}\right)(\beta f_{,r} + \mu f_{,rt} + \alpha_1 f_{,rtt}) = r f_{,tt}, \tag{8}$$

where we have dropped the superimposed bar. In equations (7), ρ_0 equals the mass density of the viscoelastic material in the reference configuration and β_0 the shear modulus for a typical viscoelastic material. Henceforth, unless otherwise specified, we use non-dimensional variables.

We write the time-harmonic displacement field as

$$f(r, t) = \mathrm{Re}\,\{\Xi(r)\mathrm{e}^{\mathrm{i}\omega t}\}, \tag{9}$$

where $\Xi(r)$ may be a complex function. Thus, there may be a phase difference between f and the applied tangential displacement.

Substituting from equation (9) into equation (8) yields

$$r\Xi'' + 3\Xi' - \frac{r\omega^2\Xi}{\beta - \alpha_1\omega^2 + \mathrm{i}\mu\omega} = 0, \tag{10}$$

where $\Xi' = \mathrm{d}\Xi/\mathrm{d}r$. A solution of equation (10) is

$$\Xi(r) = \frac{1}{r}\{c_1 I_1(rx)) + c_2 K_1(rx)\}, \tag{11}$$

where constants c_1 and c_2 are determined by the boundary conditions, $I_1(\cdot)$ and $K_1(\cdot)$ are modified Bessel functions of the first and second kind, respectively, and $x = \sqrt{\omega^2/(-\beta - \mathrm{i}\mu\omega + \alpha_1\omega^2)}$.

Writing $\Xi(r) = p(r) + \mathrm{i}q(r)$, where p and q are, respectively, the real and imaginary parts of Ξ, we obtain

$$f(r, t) = p(r)\cos(\omega t) - q(r)\sin(\omega t) = \sqrt{p^2 + q^2}\cos(\omega t + \delta), \tag{12}$$

where $\tan\delta = q/p$. Substituting from equation (12) into equation (5) and the result into equation (4) gives expressions for $\mathbf{B}$, $\mathbf{A}_1$, $\mathbf{A}_2$ and $T_{r\theta}$. The energy dissipated, per unit length of the cylinder, during a cycle of deformation equals

$$\Psi = \int_0^{2\pi/\omega}\int_1^{r_2} 4\pi r T^v_{r\theta}(r, t) D_{r\theta}(r, t)\,\mathrm{d}r\,\mathrm{d}t = 2\pi^2\mu\omega\int_1^{r_2} r^3(p'^2 + q'^2)\,\mathrm{d}r. \tag{13}$$

Besides the explicit dependence of Ψ upon μ and ω as exhibited in equation $(13)_2$, the energy dissipation also depends upon β and α_1 through the dependence of p and q on μ, ω, β and α_1.

The moment, M, per unit length of a cylinder of radius r is given by

$$\begin{aligned} M &= 2\pi r^2 T_{r\theta}(r, t) \\ &= 2\pi r^3[(\beta - \alpha_1\omega^2)(p'\cos\omega t - q'\sin\omega t) - \mu\omega(p'\sin\omega t + q'\cos\omega t)], \end{aligned} \tag{14}$$

and the work, W, during one cycle of deformation done per unit cylinder length by external forces applied to the outer cylinder is found to be

$$W = \int_0^{2\pi/\omega} M(r_2, t)(-\Lambda_0 \omega \sin \omega t)\,\mathrm{d}t$$
$$= 2\pi^2 r_2^3 \Lambda_0 [\beta q' + \mu\omega p' - \alpha_1 \omega^2 q']. \tag{15}$$

Thus, W depends explicitly upon values of all three material parameters, and also implicitly upon them through the dependence of p and q upon β, μ and α_1. The changes in the kinetic energy and the energy stored per unit length of the viscoelastic cylinder over a cycle of deformation vanish because $f(r, t)$ is a harmonic function of time t. Hence, the energy balance gives $W = \Psi$.

In the absence of inertia effects, $f_{,tt} = 0$, the deformation field is

$$\hat{f}(r, t) = \frac{\Lambda_0 r_2^2}{1 - r_2^2}\left(\frac{1}{r^2} - 1\right)\cos \omega t. \tag{16}$$

That is, the deformation field for very slow deformations of the viscoelastic material is independent of the values of material parameters; this is because displacements are prescribed on the inner and outer surfaces of the viscoelastic layer. The angular displacement of every material point is in phase with the tangential displacement prescribed on the outer cylinder, and $\delta = 0$. Here and below, we denote quantities for the inertialess problem by a superposed hat. Yu and Batra [5] have scrutinized the damping in a viscoelastic layer enclosed between a rigid cylinder and a hollow rubber cylinder. They neglected the effect of inertia forces and modeled the viscoelastic layer by a history type constitutive relation.

The energy dissipated and the work done by external forces over a cycle of deformation, and the moment applied on the outer cylinder per unit cylinder length under the assumption of negligible inertia forces are

$$\hat{W} = \hat{\Psi} = \frac{4\Lambda_0^2 \pi^2 r_2^2 \mu\omega}{r_2^2 - 1}, \quad \hat{M} = \frac{4\pi\Lambda_0 r_2(\beta\cos(\omega t) - \mu\omega\sin(\omega t))}{r_2^2 - 1}. \tag{17}$$

Thus, the energy dissipated per cycle of deformation depends linearly upon the Newtonian viscosity μ, and the frequency of the applied tangential displacement. However, the moment required to twist the outer cylinder also depends upon the elastic modulus of the viscoelastic layer.

We delineate below the effect on the deformation field and the energy dissipated of various material parameters, the frequency ω of the displacement field prescribed on the outer surface of the cylinder, and the radii r_1 and r_2 of the inner and outer cylinders. The reference values of non-dimensional variables used in the computation of results are $\beta = 1$, $\omega = 3$, $\mu = 0{\cdot}1$, $\alpha_1 = 0{\cdot}03$, $r_2 = 2$, $\Lambda_0 = 1$, $\rho = 1$. Values of the corresponding dimensional and the other reference variables are $\beta_0 = 0{\cdot}35$ MPa, $\rho_0 = 1200\ \mathrm{kg/m^3}$, $\mu = 40{\cdot}98$ Pa s, $\tau = 11{\cdot}71$ ms, $r_1 = 0{\cdot}2$ m, $r_2 = 0{\cdot}4$ m, $\alpha_1 = 0{\cdot}4799\ \mathrm{Pa\,s^2}$.

These values of material parameters, except possibly those of μ and α_1 are for a rubber-like material. Except for the non-dimensional thickness, we study the effect of varying each parameter over a large range.

Figures 1(a)–1(d) exhibit for different values of ω, μ, α_1 and β, the variation through the thickness of the magnitude, $|f|$, or $(p^2 + q^2)^{1/2}$, of the angular displacement f; its value for negligible inertia effects is plotted in Figure 1(a). Recall that the latter is independent of the

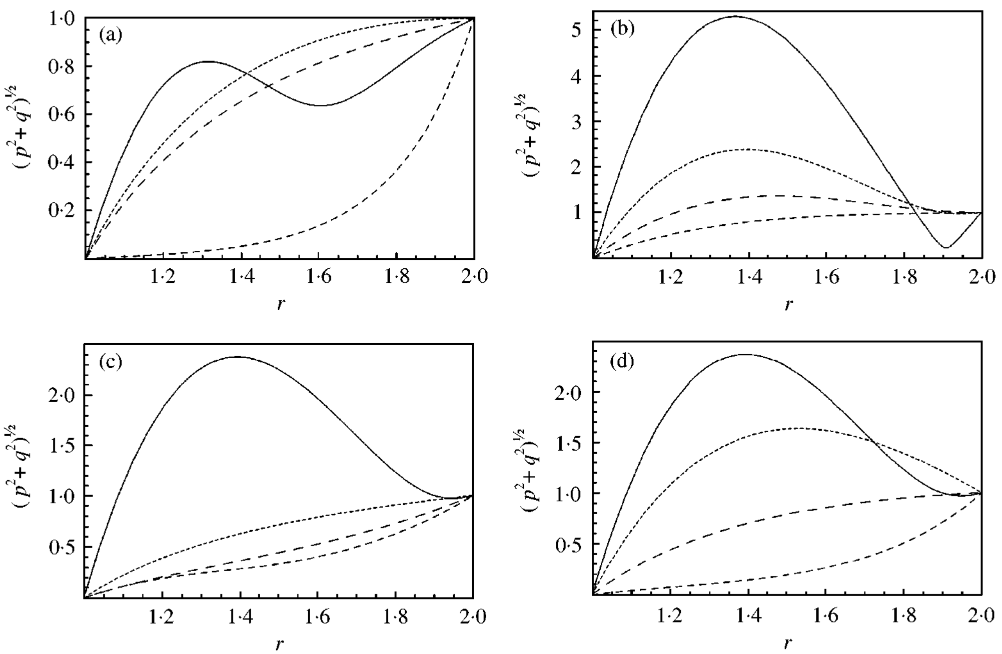

Figure 1. Through-the-thickness variation of the magnitude of the angular displacement, $|f|$ or $(p^2+q^2)^{1/2}$, for (a) five values of the frequency ω: (— — —), $\omega = 0{\cdot}0$ and $0{\cdot}04$; (- - - - -), $\omega = 1$; (——), $\omega = 4$; (– – – –), $\omega = 40$. (b) four values of the Newtonian viscosity μ: (——), $\mu = 0{\cdot}01$; (- - - - -), $\mu = 0{\cdot}1$; (— — —), $\mu = 0{\cdot}2$; (– – – –) $\mu = 0{\cdot}5$; (c) four values of the higher order viscosity α_1: (——) $\alpha_1 = 0{\cdot}03$; (– – – –), $\alpha_1 = 0{\cdot}1$; (— — —), $\alpha_1 = 0{\cdot}3$; (- - - - -), $\alpha_1 = 3$; and (d) four values of the elastic modulus β: (——) $\beta = 0{\cdot}2$; (- - - - -) $\beta = 1$; (— — —), $\beta = 2$; (– – – –) $\beta = 20$.

values of ω, μ, α_1 and β. It is clear from the plots of Figure 1(a) that for $\omega = 0{\cdot}04$, the variation of $|f|$ through the thickness of the viscoelastic layer is essentially the same as for the inertialess problem. With an increase in the frequency of the applied angular displacement and hence of the effect of inertia forces, the variation of $|f|$ versus r changes from concave downwards to concave upwards. For $\omega = 4$, the curve consists of two parts-concave downwards near the periphery of the inner cylinder and concave upwards at points close to the outer cylinder. The plots in Figure 1(b) reveal that for very low values of the Newtonian viscosity, μ, the variation of $|f|$ is not monotonic through the thickness of the viscoelastic layer; the maximum value, $|f|_m$, of $|f|$ occurs at a point within the layer and $|f|_m$ decreases with an increase in the value of μ till for $\mu = 0{\cdot}5$, $|f|_m$ occurs at the outermost surface. For $0{\cdot}1 \leqslant \mu \leqslant 0{\cdot}5$, the variation of $|f|$ with r is very gradual at points close to the outer cylinder. Results plotted in Figure 1(c) suggest that very low values of the higher order viscosity, α_1, have a negligible influence on the through-thickness variation of $|f|$. However, as α_1 is increased from 0·1 to 3, the graph of $|f|$ versus r changes from concave upwards to concave downwards. Higher values of the elastic shear modulus, β, change the profile of $|f|$ versus r from concave downwards to concave upwards, and reduce the maximum value of $|f|$.

The through-thickness variation of the phase-difference angle, δ, for different values of ω, μ, α_1 and β is exhibited in Figures 2(a)–2(d); note that $\delta = 0$ when the effect of inertia forces is negligible. The magnitude of δ is maximum at $r = 1$ and monotonically decreases to zero at $r = 2$ for low ($\omega \simeq 2$) and high ($\omega \simeq 10$) frequencies, large values of the Newtonian viscosity ($\mu \geqslant 0{\cdot}2$), small ($\alpha_1 \simeq 0{\cdot}01$) and large ($\alpha_1 \simeq 0{\cdot}2$) values of the higher order viscosity α_1 and large values of the elastic shear modulus β. There is a range of values of ω, μ, α_1 and

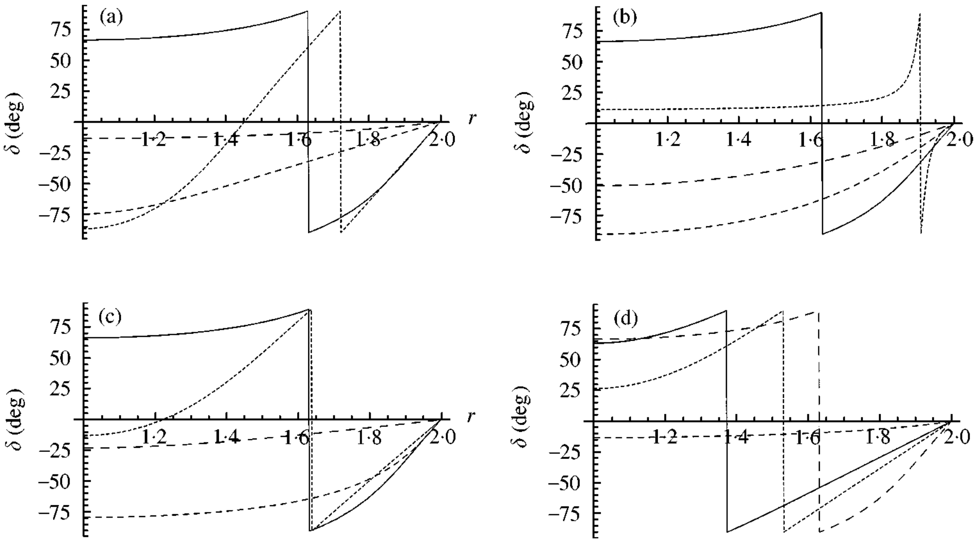

Figure 2. Through-the-thickness variation of the phase angle, δ for (a) four values of the frequency ω: (— — —), $\omega = 2$; (——), $\omega = 3$; (- - - - -), $\omega = 5$; (– – – –), $\omega = 10$; (b) four values of the Newtonian viscosity μ: (- - - - -), $\mu = 0{\cdot}01$; (——), $\mu = 0{\cdot}1$; (– – – –), $\mu = 0{\cdot}2$; (— — —), $\mu = 0{\cdot}5$; (c) four values of the higher order viscosity α_1: (– – – –), $\alpha_1 = 0{\cdot}01$; (——), $\alpha_1 = 0{\cdot}03$; (- - - - -), $\alpha_1 = 0{\cdot}2$; (— — —), $\alpha_1 = 0{\cdot}2$; (d) four values of the elastic modulus: (——), $\beta = 0{\cdot}1$; (- - - - -), $\beta = 0{\cdot}2$; (— — —), $\beta = 1$; (– – – –), $\beta = 2$.

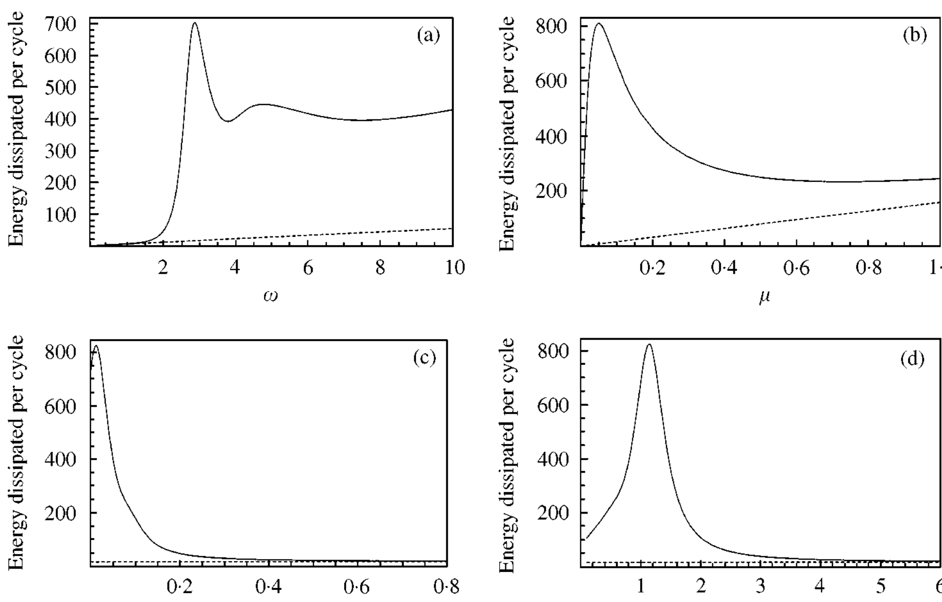

Figure 3. The dependence of the energy dissipated per cycle per unit cylinder length, Ψ, upon (a) the frequency ω, (b) the Newtonain viscosity μ, (c) the higher order viscosity α_1 and (d) the elastic modulus β: (——), with inertia; (- - - - -), without inertia.

β for which the variation of δ versus r is not monotonic and its value suddenly jumps from a large positive value to a large negative value across the surface $r = r_c$, $1 < r_c < 2$.

The dependence of the energy dissipated per cycle per unit length of the cylinder, Ψ, upon ω, μ, α_1 and β is plotted in Figures 3(a)–3(d). The energy dissipated first increases almost linearly with an increase in the value of ω from 0 to 1·5 and then increases quite rapidly, attains a maximum at $\omega \simeq 3$ rad/s and is almost constant for $\omega \geq 6$ rad/s; a somewhat similar pattern is observed when other parameters are varied. The energy dissipated stays essentially unchanged for $\mu \geq 0{\cdot}5$, $\alpha_1 \geqslant 0{\cdot}5$ and $\beta \geqslant 4$. Thus, higher values of the Newtonian viscosity or the higher order viscosity beyond a certain value do not increase the energy dissipated per cycle. However, when the effect of inertia forces is neglected, Ψ increases linearly with the Newtonain viscosity and does not depend upon α_1 and β. Thus, the solution of the inertialess problem does not provide even qualitatively a correct dependence of the energy dissipated upon different material variables.

ACKNOWLEDGMENTS

This work was supported by the ARO grant DAAG 55-98-1-0030 to Virginia Polytechnic Institute and State University.

REFERENCES

1. R. L. FOSDICK, W. H. WARNER and J. H. YU 1990 *International Journal of Engineering Science* **28**, 469–483. Steady, structured shock wave in a viscoelastic solid of differential type.
2. R. L. FOSDICK and J. H. YU 1996 *International Journal of Non-linear Mechanics* **31**, 495–516. Thermodynamics, stability and non-linear oscillations of viscoelastic solids — I. Differential type solids of second grade.
3. H. MARKOVITZ and B. D. COLEMAN 1964 *The Physics of Fluids* **7**, 833–841. Nonsteady helical flows of second-order fluids.
4. C. TRUESDELL and W. NOLL 1965 *Handbuch der Physik* III/3, New York: Springer. The non-linear field theories of mechanics.
5. J. H. YU and R. C. BATRA 2000 *Journal of Sound and Vibration* **229**, 445–452. Constrained layer damping in finite shearing deformations.