STABILITY OF A PROPAGATING INTERPHASE BOUNDARY IN A THERMOPLASTIC MATERIAL

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We study the morphological stability of a propagating planar interphase boundary in a thermoplastic material deformed in antiplane shear. The plane interphase boundary is found to be stable if $\Sigma_0 / (4\rho c) > V_0^2$ where c is the specific heat per unit volume; ρ is the mass density; V_0 is the propagation speed; and Σ_0 is a function of the material moduli and the state of deformation of the body.

The propagation of interphase boundaries in one- and two-dimensional deformations of solids that can undergo phase transformations has been studied by, amongst others, James [12], Hutchinson and Neale [11], Coleman [6], Fager and Bassani [7], and Tugcu and Neale [20, 21]. Many of these studies have been motivated by the experimental observations on the cold-drawing of polymer fibers and membranes. These works have examined the existence and properties of a steady-state solution characterized by a steadily propagating interphase boundary that divides the body into two uniform or nearly uniform deformed regions. This phenomenon is somewhat similar to the directional solidification process studied by Mullins and Sekerka [18], Langer [16], and Godreche [10] where a moving planar interface separates the solid and liquid regions. For this case a planar interface can become morphologically unstable and then develop into a cellular or dendritic pattern. We note that detailed experimental observations on the morphology of a propagating interphase boundary in two-dimensional deformations of solids undergoing phase transformations has not been reported. Here we use Mullins and Sekerka's method to study analytically the morphological stability of a planar propagating interphase boundary in phase-transforming thermoplastic solids deformed in antiplane shear. For a thermoelastic solid deformed in antiplane shear, Fried [9] has investigated the relationship between the morphological stability of the planar interface and the "kinetic functions" proposed by Knowles [15] and Abeyaratne and Knowles [1].

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FORMULATION OF THE PROBLEM

We consider antiplane shear deformations in the $\{x_1, x_2\}$-plane of a rigid-plastic and nonheat conducting material whose behavior can be described by the J_2-deformation theory. We confine ourselves to the geometrically linearized case; therefore, the effect of change of configuration during phase transition is neglected. We also ignore the effect of thermal expansion and unloading of a material point. The J_2-deformation theory for thermoplastic materials has been used in the analysis of many problems (e.g., see Boley and Weiner [5], Baines [4], and Kachanov [14]). In it the stress–strain relation is given by

$$s_{ij} = (2/3)(\sigma_e/\epsilon_e)\bar{\epsilon}_{ij}$$

where s_{ij} and $\bar{\epsilon}_{ij}$ respectively, are the components of the deviatoric stress tensor and the deviatoric infinitesimal strain tensor and

$$\sigma_e = (2s_{ij}s_{ij}/3)^{1/2} \quad \text{and} \quad \epsilon_e = \left(2\bar{\epsilon}_{ij}\bar{\epsilon}_{ij}/3\right)^{1/2}$$

are the effective stress and the effective strain, respectively. It can be verified that for the J_2-deformation theory there exists a plastic work function $W(\epsilon_e)$ such that

$$s_{ij} = \partial W/\partial\bar{\epsilon}_{ij}$$

For the antiplane shear deformations in this article, let the nonzero displacement component along the vertical direction be $w(x_1, x_2, t)$. From the linearized strain-displacement equations, two nonzero components of the strain tensor ϵ_{ij} are

$$\epsilon_{13} = w_{,1}/2 \qquad \epsilon_{23} = w_{,2}/2 \tag{1}$$

where $w_{,i} = \partial w/\partial x_i$. Here we assume that a potential function $W(\epsilon_{13}, \epsilon_{23}, \theta)$ per unit volume exists and is of a separable form

$$W(\epsilon_{13}, \epsilon_{23}, \theta) = F(\Pi)I(\theta) \tag{2}$$

where $\theta = T - T_0$, T_0 is a suitably chosen reference value of the temperature T, and Π is proportional to the effective strain ϵ_e and is defined as

$$\Pi = (\epsilon_{13}^2 + \epsilon_{23}^2)^{1/2} \tag{3}$$

for convenience. We note that several authors, e.g., see Litonski [17] and Johnson and Cook [13] have assumed that the flaw stress can be expressed as a product of three functions: one of temperature alone, another of effective plastic strain-rate, and the third one of effective plastic strain. In order that the material exhibit thermal softening, we require that (i) $I(0) = 1$ and $0 <$ a positive number $\leq I(\theta) \leq 1$ for the range of $\theta \geq 0$ to be discussed herein, and (ii) $I'(\theta) < 0$. The function $F(\cdot)$ is assumed to be a nonlinear function of Π such that

$$f(\Pi) = F_{,\Pi}(\Pi)/2 = \frac{1}{2}\frac{dF}{d\Pi} \tag{4}$$

is characterized by the "rising-falling-rising" curve as shown in Figure 1. It follows from Eq. (2) that components σ_{13} and σ_{23} of the stress tensor σ_{ij} are given by

$$\begin{aligned} \sigma_{13} &= (1/2)\partial W/\partial\epsilon_{13} = f(\Pi)I(\theta)\epsilon_{13}/\Pi \\ \sigma_{23} &= (1/2)\partial W/\partial\epsilon_{23} = f(\Pi)I(\theta)\epsilon_{23}/\Pi \end{aligned} \tag{5}$$

respectively; thus the effective stress is proportional to $f(\Pi)I(\theta)$. Therefore, by using the uniaxial stress–strain curve shown in Figure 1, a relation between the effective stress and the effective strain at any given temperature can be derived.

For the problem being studied, the balance of linear momentum and the balance of internal energy are equivalent to the equations

$$\rho w = \sigma_{13} + \sigma_{23,2} \tag{6}$$

$$c\theta_{,t} = \sigma_{13}w_{,1t} + \sigma_{23}w_{,2t} \tag{7}$$

respectively, where ρ is the constant mass density and c is the constant specific heat per unit volume. In Eq. (7) we have assumed that all of the plastic working, rather than 90% to 95% of it as asserted by Farren and Taylor [8] and Suli-

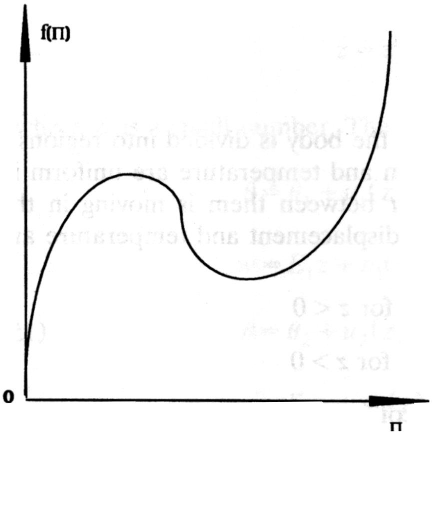

Figure 1. The uniaxial constitutive function $f(\Pi)$

joadikusumo and Dillon [19] is converted into heating. From Eqs. (6) and (7), we obtain the balance of total energy

$$(c\theta + \rho w_{,t}^2/2)_{,t} = (\sigma_{13} w_{,t})_{,1} + (\sigma_{23} w_{,t})_{,2} \tag{8}$$

and recalling Eq. (5), we write Eq. (7) as

$$cI^*(\theta)_{,t} = F \tag{9}$$

where

$$I^*(\theta) = \int_0^\theta I(\zeta)^{-1}\, d\zeta$$

Obviously $I^*(\theta)$ is an increasing function of θ.

Since some of the physical quantities will be discontinuous across the interface, we need jump conditions across it. Let the interface propagate at the local speed $\mathbf{V}$ and its local normal vector be $\mathbf{n}$ (directed in the direction of propagation); the jump conditions across the interface are

$$[\![w]\!] = 0 \tag{11}$$

$$-\rho V_n [\![w_{,t}]\!] = [\![\boldsymbol{\sigma}]\!] \cdot \mathbf{n}$$

$$c [\![I^*(\theta)]\!] = [\![F]\!]$$

$$-V_n [\![c\theta + \rho (w_{,t})^2/2]\!] = [\![w_{,t} \boldsymbol{\sigma}]\!] \cdot \mathbf{n} \tag{14}$$

where $[\![\cdot]\!] = (\cdot)^+ - (\cdot)^-$ denotes the jump, $\boldsymbol{\sigma} = \{\sigma_{13}, \sigma_{23}\}$, and $V_n = \mathbf{V} \cdot \mathbf{n} \geq 0$ is the normal component of the velocity $\mathbf{V}$ of propagation. The jump condition (13) is obtained from Eq. (9), and others are standard ones; for example, see Abeyaratne and Knowles [2]. Whereas Fried [9] used "kinetic relations," we employ jump conditions (12) to (14) derived from the conservation laws.

A STEADY-STATE SOLUTION

We examine the steady-state solution for which the body is divided into regions I and II, as shown in Figure 2, such that the strain and temperature are uniform in these regions and the planar interface $x_1 = V_0 t$ between them is moving in the x_1-direction at a constant velocity V_0. Thus the displacement and temperature are given by

$$\begin{aligned} \theta &= \theta_1 \qquad w = E_1 z \qquad \text{for } z < 0 \\ \theta &= \theta_2 \qquad w = E_2 z \qquad \text{for } z > 0 \end{aligned} \tag{15}$$

where $z = x_1 - V_0 t$ and θ_i and E_i ($i = 1, 2$) are constants with $\theta_2 < \theta_1$ and $E_2 < E_1$

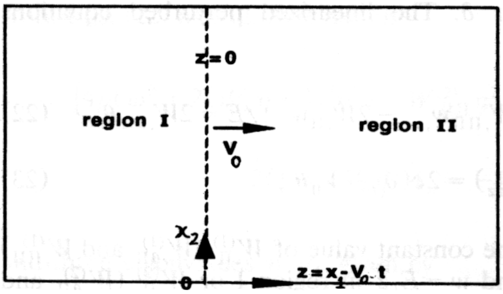

Figure 2. Propagating interface.

The jump conditions (12) to (14) reduce to

$$\begin{aligned}
&\rho V_0^2(E_1-E_2)=f(E_1)I(\theta_1)-f(E_2)I(\theta_2)\\
&c(I^*(\theta_1)-I^*(\theta_2))=F(E_1)-F(E_2)\\
&c(\theta_1-\theta_2)+\rho V_0^2(E_1^2-E_2^2)/2=f(E_1)E_1I(\theta_1)-f(E_2)E_2I(\theta_2)
\end{aligned}\tag{17}$$

It is seen that the deformation and temperature are coupled with each other.

ANALYSIS OF THE MORPHOLOGICAL STABILITY OF THE INTERFACE

Governing Equations

Following Mullins and Sekerka's method (Mullins and Sekerka [18], Langer [16], and Godreche [10]) we consider a perturbation of the interface geometry in the reference frame $\{z; x_2; t\}$. Let the interface geometry be given an infinitesimal perturbation

$$z=z^*(x_2,t)=\delta\cos kx_2e^{\omega t}$$

where δ is a small number. The steady-state solution (15) is replaced by

$$\begin{aligned}
&\theta=\theta_1+u_1(z)\cos kx_2e^{\omega t}\\
&w=E_1z+v_1(z)\cos kx_2e^{\omega t}\quad \text{for } z<z^*(x_2,t)
\end{aligned}\tag{20}$$

$$\begin{aligned}
&\theta=\theta_2+u_2(z)\cos kx_2e^{\omega t}\\
&w=E_2z+v_2(z)\cos kx_2e^{\omega t}\quad \text{for } z>z^*(x_2,t)
\end{aligned}\tag{21}$$

where u_1, v_1 and u_2, v_2 are of order δ. The linearized perturbed equations obtained from Eqs. (6) and (7) are

$$4\rho(w_{,tt} - 2V_0 w_{,zt} + V_0^2 w_{,zz}) - W_{,\Pi\Pi} w_{,zz} - 2W_{,\Pi} w_{,zz}/E = 2W_{,\theta\Pi}\theta_{,z}$$

$$W_{,\Pi}(w_{,zt} - V_0 w_{,zz}) = 2c(\theta_{,t} - V_0\theta_{,z})$$

where $W_{,\Pi}$ ($W_{,\theta\Pi}$ and $W_{,\Pi\Pi}$) denotes the constant value of $W_{,\Pi}^{(1)}$ ($W_{,\theta\Pi}^{(1)}$ and $W_{,\Pi\Pi}^{(1)}$) at the uniform state $\theta = \theta_1$, $E = E_1$, and $w = E_1 z$ in region I or $W_{,\Pi}^{(2)}$ ($W_{,\theta\Pi}^{(2)}$ and $W_{,\Pi\Pi}^{(2)}$) at the uniform state $\theta = \theta_2$, $E = E_2$, and $w = E_2 z$ in region II. Equations (22) and (23) are coupled unless $W_{,\Pi} = 0$ or $W_{,\Pi\theta} = 0$.

In order to find a solution of Eqs. (22) and (23), we first examine their eigensolutions. Let

$$u_1(z) \sim A_\lambda e^{\lambda z} \quad v_1(z) \sim B_\lambda e^{\lambda z} \qquad \text{for } z < z^*(x_2, t$$

where A_λ and B_λ are the λ-dependent coefficients; then Eqs. (22) and (23) yield

$$\{4\rho(\omega^2 - 2\omega V_0\lambda + V_0^2\lambda^2) - W_{,\Pi\Pi}^{(1)}\lambda^2 + 2W_{,\Pi}^{(1)}k^2/E_1\}B_\lambda = 2W_{,\theta\Pi}^{(1)}\lambda A$$

$$2c(\omega - V_0\lambda)A_\lambda = W_{,\Pi}^{(1)}(\omega - V_0\lambda)\lambda B_\lambda$$

and three eigenvalues λ are solutions of

$$c\{4\rho(\omega^2 \quad 2\omega V_0\lambda + V_0^2\lambda^2) - W_{,\Pi\Pi}^{(1)}\lambda^2 + 2W_{,\Pi}^{(1)}k^2/E_1\} = W_{,\Pi}^{(1)}W_{,\theta\Pi}^{(1)}\lambda^2$$

$$\omega = V_0\lambda$$

and the ratio of coefficients A_λ to B_λ of the eigensolution is given by either Eq (25) or (26).

Equations (27) give

$$\lambda = \quad -4c\rho\omega V_0 \pm \sqrt{4c\rho\omega^2\Sigma_0^{(1)} + 2cW_{,\Pi}^{(1)}k^2\Sigma^{(1)}/E_1}\,\Big)/\Sigma^{(1)} \tag{28}$$

where

$$\Sigma \equiv \Sigma_0 - 4c\rho V_0^2 \tag{29}$$

$$\Sigma_0 \equiv cW_{,\Pi\Pi} + W_{,\Pi}W_{,\theta\Pi} \tag{30}$$

and the superscript (1) on Σ_0 and Σ signifies their values in region Similarly, let

$$u_2(z) \sim A_\eta e^{-\eta z} \quad v_2(z) \sim B_\eta e^{-\eta z} \qquad \text{for } z > z^*(x_2, t) \tag{31}$$

from Eqs. (22) and (23) we obtain

$$\{4\rho(\omega^2 + 2\omega V_0\eta + V_0^2\eta^2 \; - W^{(2)}_{,\Pi\Pi}\eta^2 + 2W^{(2)}_{,\Pi}k^2/E_2\}B_\eta = \quad 2W^{(2)}_{,\theta\Pi}\eta A_\eta \quad (32)$$

$$2c(\omega + V_0\eta)A_\eta = -W^{(2)}_{,\Pi}(\omega + V_0\eta)\eta B_\eta \quad (33)$$

and three eigenvalues η are given by

$$c\{4\rho(\omega^2 + 2\omega V_0\eta + V_0^2\eta^2) - W^{(2)}_{,\Pi\Pi}\eta^2 + 2W^{(2)}_{,\Pi}k^2/E_2\} = W^{(2)}_{,\Pi}W^{(2)}_{,\theta\Pi}\eta^2$$

$$\omega + V_0\eta = 0$$

The ratio of coefficients A_η to B_η of the eigensolution is given by either Eq. (32) or (33). Two roots of Eqs. $(34)_1$ are

$$\eta = \left(4c\rho\omega V_0 \pm \sqrt{4c\rho\omega^2\Sigma_0^{(2)} + 2cW^{(2)}_{,\Pi}k^2\Sigma^{(2)}/E_2}\right)/\Sigma^{(2)}$$

We note that there are six roots of eigenequations (27) and (34). Because boundary conditions at infinity are not perturbed, the admissible perturbation must tend to zero at infinity. Thus real parts of admissible roots must be positive. Then, according to the Mullins and Sekerka method, it is essential for our present problem that there be three and only three admissible roots of Eqs. (27) and (34) with positive real parts such that the corresponding three unknown coefficients and δ may be well-determined by the four interface jump conditions (11) to (14). We assume that the roots p, r, and q exist and require that

$$\mathrm{Re}(p) > 0 \qquad \mathrm{Re}(r) > 0 \qquad \mathrm{Re}(q) > 0$$

If the boundary conditions at infinity are also given, then the infinitesimal perturbations and inequalities (36) are relaxed to

$$\mathrm{Re}(p) \geq 0 \qquad \mathrm{Re}(r) \geq 0 \qquad \mathrm{Re}(q) \geq 0$$

and we shall have a more restrictive definition of stability because the set of admissible perturbations is enlarged. We do not study this alternative.

Because of the assumption of infinitesimal deformations and infinitesimal perturbations, the local velocity

$$V = V_0 + z^*_{,t}(x_2, t)$$

of the interface nearly equals its normal component V_n. Thus, in the frame $\{z, x_2, t\}$, four jump conditions (11) to (14) at $z = z^*(x_2, t)$ reduce to the following four linear homogeneous equations

$$\delta(E_1 - E_2) + [\![V]\!] = 0$$

$$4\rho\omega V_0 \delta(E_1 - E_2) + [\![(2\rho V_0^2 - W^0_{.\Pi\Pi}/2) v_{.z}]\!] = [\![W^0_{.\Pi\theta} u]\!]$$

$$V_0 [\![(E W^0_{,\Pi\Pi}/2 - 2\rho V_0^2 E + 2 f_0 I(\theta)) v_{,z}]\!]$$

$$= 2\omega\delta [\![f_0 I^0(\theta) E]\!] + V_0 [\![(2c - E W^0_{,\Pi\theta}) u]\!] + 2\omega [\![(f_0 I(\theta) \quad \rho V_0^2 E) v]\!]$$

$$[\![F^0_{,\Pi} v_{.z}]\!] = 2c [\![u / I^0(\theta)]\!]$$

where $u = u_1, u_2$ and $E = E_1, E_2$ for regions I, II, respectively.

From the condition for the existence of a nonzero solution $\{\delta, u, v\}$ we get an expression for ω and then examine the interface stability.

Stability Conditions

We delineate all possible unstable cases; thus, the rest must be stable. For the interface to be unstable, $\mathrm{Re}(\omega) > 0$. Relations (27), (28), (34), and (35) imply that there cannot be five or six admissible roots with positive real parts. If there are four admissible roots, then, since there are four corresponding arbitrary coefficients, four jump conditions may always be satisfied by these four roots and δ with $\mathrm{Re}(\omega) > 0$. Therefore, the interface will be unstable. In fact, this is the case when one of the two values of $\Sigma^{(i)}$ in two distinct phases is negative, that is, when $\Sigma^{(1)} < 0$ and $\Sigma^{(2)} > 0$. However, this is not the case when $\Sigma^{(2)} > \Sigma^{(1)} > 0$; in other words, the possibility of four admissible roots of Eqs. (27) and (34) with $\mathrm{Re}(\omega) > 0$ is excluded by $\Sigma^{(2)} > \Sigma^{(1)} > 0$. Henceforth we assume that $\Sigma^{(2)} > \Sigma^{(1)}$.

For the general case when there are only three admissible roots of Eqs. (27), (28), (34), and (35), the following are the only three possibilities for $\mathrm{Re}(\omega) > 0$.

Case 1. There is no admissible root of Eq. (34), and there are three admissible roots of Eq. (27);

Case 2. There is one admissible root of Eq. (34), and two admissible roots of Eq. (27);

Case 3. There are two admissible roots of Eq. (34) and just one admissible root of Eq. (27).

Case 3 is impossible for real ω^2. Below we examine each one of these three cases.

Case 1. Three roots p, r, q are given by Eqs. (28) and $(27)_2$ and we have

$$u_1(z) = \alpha e^{pz} + \psi e^{rz} + \beta^* e^{qz}$$

$$v_1(z) = \alpha^* e^{pz} + \psi^* e^{rz} + \beta e^{qz} \qquad z < z^*(x_2, t)$$

$$u_2(z) = 0 \quad v_2(z) = 0 \qquad z > z^*(x_2, t)$$

where α, β, ψ are three unknown constants and $\alpha^*, \beta^*, \psi^*$ are expressed in terms of α, β, ψ through Eqs. (25) and (26) as

$$\{-W^{(1)}_{,\Pi\Pi} q^2 + 2W^{(1)}_{,\Pi} k^2/E_1\}\beta = 2W^{(1)}_{,\theta\Pi} q\beta^*$$

$$2c\alpha = W^{(1)}_{,\Pi} p\alpha^* \tag{43}$$

$$2c\psi = W^{(1)}_{,\Pi} r\psi^*$$

The four unknown constants $\delta, \alpha, \beta, \psi$ will be determined by the four interface conditions (38) to (41). Then, we get the following three equations for $v_{1,z}$, u_1, and v_1:

$$(2\rho V_0^2 - W^{(1)}_{,\Pi\Pi}/2)v_{1,z} - W^{(1)}_{,\Pi\theta} u_1 = 4\rho\omega V_0 v$$

$$(W^{(1)}_{,\Pi\Pi}/2 - 2\rho V_0^2)v_{1,z} + 2\rho V_0\omega(E_1 + E_2)v_1/E_1 + W^{(1)}_{,\Pi\theta} u_1 = 0$$

$$W^{(1)}_{,\Pi} v_{1,z} = 2cu_1 \tag{46}$$

Since $E_1 \neq E_2$, we conclude from Eqs. (44) and (45) that

$$v_1 = 0$$

and furthermore from Eq. (44) we obtain

$$(2\rho V_0^2 - W^{(1)}_{,\Pi\Pi}/2)v_{1,z} = W^{(1)}_{,\Pi\theta} u_1 \tag{48}$$

Therefore $\Sigma^{(1)} \neq 0$ is equivalent to

$$v_{1,z} = v_1 = u_1 = 0 \tag{49}$$

or

$$\alpha + \phi + \beta^* = 0 \qquad \alpha^* + \phi^* + \beta = 0 \qquad p\alpha^* + r\phi^* + q\beta = 0$$

or, more explicitly,

$$W^{(1)}_{,\Pi}(r-p)\alpha^* + W^{(1)}_{,\Pi} r\beta - 2c\beta^* = 0 \qquad (r-p)\alpha^* + (r-q)\beta = 0$$

It follows that

$$\Sigma_0^{(1)}\omega^2 = 2cV_0^2 W_{,\Pi}^{(1)} k^2/E_1 \tag{52}$$

Therefore, for the existence of a solution with $\mathrm{Re}(\omega) > 0$, we must have $\Sigma_0^{(1)} > 0$; the corresponding roots given by Eqs. (28) and $(27)_2$ are all possible admissible roots if and only if $\Sigma^{(1)} < 0$ and $\Sigma^{(2)} < 0$. Thus we conclude that when and only when $\Sigma^{(1)} < 0$ and $\Sigma^{(2)} < 0$, there can exist a solution with $\omega > 0$, and then the steady-state solution (15) with the interface $z = 0$ is unstable.

Case 2. We write the admissible roots as p, r, and q with

$$\omega = V_0 p$$

$$r = \quad -4c\rho\omega V_0 + \sqrt{4c\rho\omega^2\Sigma_0^{(1)} + 2cW_{,\Pi}^{(1)}k^2\Sigma^{(1)}/E_1}\Big)/\Sigma^{(1)}$$

$$q = \Big(4c\rho\omega V_0 + \sqrt{4c\rho\omega^2\Sigma_0^{(2)} + 2cW_{,\Pi}^{(2)}k^2\Sigma^{(2)}/E_2}\Big)/\Sigma^{(2)}$$

The general forms of admissible perturbations are

$$\begin{aligned} &u_1(z) = \alpha^* e^{pz} + \psi e^{rz} \quad v_1(z) = \alpha e^{pz} + \psi^* e^{rz} \qquad z < z^*(x_2, t) \\ &u_2(z) = \beta e^{-qz} \quad v_2(z) = \beta^* e^{-qz} \qquad z > z^*(x_2, t) \end{aligned} \tag{54}$$

where α, β, ψ are three unknown constants and $\alpha^*, \beta^*, \psi^*$ are expressed in terms of α, β, ψ through Eqs. (25), (26), (32), and (33) as

$$-W_{,\Pi\Pi}^{(1)}p^2 + 2W_{,\Pi}^{(1)}k^2/E_1\}\alpha = 2W_{,\theta\Pi}^{(1)}p\alpha^*$$

$$2c\psi = W_{,\Pi}^{(1)}r\psi^*$$

$$2c\beta = -W_{,\Pi}^{(2)}q\beta^*$$

The four unknown constants $\delta, \alpha, \beta, \psi$ will be determined from the four interface conditions (38) to (41) that now can be written as

$$W_{,\Pi\theta}^{(1)}u_1 + 2\rho\omega V_0[\![v]\!] + (W_{,\Pi\Pi}^{(1)}/2 - 2\rho V_0^2)v_{1,z} = 0$$

$$(W_{,\Pi\Pi}^{(1)}/2 - 2\rho V_0^2)v_{1,z} + (W_{,\Pi\Pi}^{(2)}/2 - 2\rho V_0^2)v_{2,z} + W_{,\Pi\theta}^{(1)}u_1 + W_{,\Pi\theta}^{(2)}u_2 = 0$$

$$W_{,\Pi}^{(1)}v_{1,z} = 2cu_1$$

Using $u_2=\beta$, $v_2=-2c\beta/qW^{(2)}_{,\Pi}$, $v_{2,z}=2c\beta/W^{(2)}_{,\Pi}$, we eliminate β and obtain from Eqs. $(56)_{1,2}$

$$W^{(1)}_{,\Pi\theta}(q\Sigma^{(2)}-4c\rho\omega V_0)u_1$$
$$+2\rho\omega V_0 q\Sigma^{(2)}v_1+(W^{(1)}_{,\Pi\Pi}/2-2\rho V_0^2)(q\Sigma^{(2)}-4c\rho\omega V_0)v_{1,z}=0$$

We assume that $v_1=u_1X+v_{1,z}Y$ and conclude that

$$2c=r\left[2cY+W^{(1)}_{,\Pi}X\right]$$

Thus the eigenequation becomes

$$\left[\Sigma^{(1)}(q\Sigma^{(2)}-4c\rho\omega V_0)r+4c\rho\omega V_0 q\Sigma^{(2)}\right]\left[\Sigma^{(1)}_0 p^2-2cW^{(1)}_{,\Pi}k^2/E_1\right]=0$$

We note that the root determined by

$$\left[\Sigma^{(1)}_0 p^2-2cW^{(1)}_{,\Pi}k^2/E_1\right]=0$$

is inadmissible. Thus

$$\Sigma^{(1)}\Sigma^{(2)}rq=4c\rho\omega V_0[r\Sigma^{(1)}-q\Sigma^{(2)}] \tag{60}$$

which with the use of Eq. (33) becomes

$$W^{(1)}_{,\Pi}W^{(2)}_{,\Pi}k^4\Sigma^{(1)}\Sigma^{(2)}+4E_1E_2(\Sigma^{(1)}_0\Sigma^{(2)}_0-16c^2\rho^2V_0^4)\rho^2\omega^4$$
$$+2\rho\omega^2k^2(\Sigma^{(2)}_0W^{(1)}_{,\Pi}\Sigma^{(1)}E_2+\Sigma^{(1)}_0W^{(2)}_{,\Pi}\Sigma^{(2)}E_1)=0$$

Thus if $\Sigma^{(2)}>\Sigma^{(1)}>0$, then $\omega^2<0$, the solution is admissible, and therefore, the interface is morphologically stable; however, if $0>\Sigma^{(2)}>\Sigma^{(1)}$, then the solution is found to be inadmissible.

Case 3. Three roots p, r, and q are expressed as

$$p=\omega/V_0$$
$$r,q=\left(4c\rho\omega V_0\pm\sqrt{4c\rho\omega^2\Sigma^{(2)}+2cW^{(2)}_{,\Pi}k^2\Sigma^{(2)}/E_2}\right)/\Sigma^{(2)} \tag{62}$$

Thus, the perturbation is given by

$$u_1(z)=\alpha^*e^{pz}\quad v_1(z)=\alpha e^{pz}\qquad \text{for } z<z^*(x_2,t)$$
$$u_2(z)=\psi e^{-rz}+\beta e^{-qz} \tag{63}$$
$$v_2(z)=\psi^*e^{-rz}+\beta^*e^{-qz}\quad \text{for } z>z^*(x_2,t)$$

where α, β, ψ are three unknown constants and $\alpha^*, \beta^*, \psi^*$ can be expressed in terms of α, β, ψ through Eqs. (25), (26), (32), and (33) as

$$-W^{(1)}_{,\Pi\Pi} p^2 + 2W^{(1)}_{,\Pi} k^2/E_1\}\alpha = 2W^{(1)}_{,\theta\Pi} p\alpha^*$$

$$2c\psi = -W^{(2)}_{,\Pi} r\psi^*$$

$$2c\beta = -W^{(2)}_{,\Pi} q\beta^*$$

The eigenequations give

$$(2\rho V_0^2 - W^{(1)}_{,\Pi\Pi}/2)v_{1,z} - W^{(1)}_{,\Pi\theta} u_1 - 4\rho\omega V_0 v_1$$

$$= (2\rho V_0^2 - W^{(2)}_{,\Pi\Pi}/2)v_{2,z} - W^{(2)}_{,\Pi\theta} u_2 - 4\rho\omega V_0 v_2 \tag{65}$$

$$(W^{(1)}_{,\Pi\Pi}/2 - 2\rho V_0^2)E_1 v_{1,z} + 2\rho V_0\omega(E_1 + E_2)v_1 + W^{(1)}_{,\Pi\theta} E_1 u$$

$$= (W^{(2)}_{,\Pi\Pi}/2 - 2\rho V_0^2)E_2 v_{2,z} + 2\rho V_0\omega(E_1 + E_2)v_2 + W^{(2)}_{,\Pi\theta} E_2 u_2$$

$$W^{(1)}_{,\Pi} v_{1,z} = 2cu_1 \tag{67}$$

Equations (67) and (63) imply that

$$2cV_0^2 W^{(1)}_{,\Pi} k^2/E_1 = \Sigma^{(1)}_0 \omega^2 \tag{68}$$

which gives a real value of ω^2 and is thus inadmissible.

If condition (68) cannot be satisfied, then $\alpha^* = \alpha = 0$; in this case $2cu_2 = W^{(2)}_{,\Pi}(v_{2,z})$, and the eigencondition reduces to

$$-8c\rho\omega V_0 v_2 = \Sigma^{(2)} v_{2,z}$$

$$-4c\rho V_0\omega(E_1 + E_2)v_2/E_2 = \Sigma^{(2)} v_{2,z}$$

Thus

$$\rho V_0\omega v_2 = 0 \quad \text{and} \quad \Sigma^{(2)} v_{2,z} = 0$$

Therefore, $v_2 = v_{2,z} = 0$ and the eigencondition becomes

$$\psi^* + \beta^* = 0 \qquad r\psi^* + q\beta^* = 0 \qquad r = q$$

which gives a real value for ω^2 also and therefore is inadmissible. Thus, Case (3) is always inadmissible.

Summary and Discussion of Results

Summarizing we conclude that

(a) If $\Sigma^{(2)} > \Sigma^{(1)} > 0$, then the unique admissible solution requires that $\omega^2 < 0$ and therefore the interface $z = 0$ is morphologically stable.
(b) If $\Sigma^{(2)} > 0$ but $\Sigma^{(1)} < 0$, then there are some admissible perturbations with $\omega > 0$ and the interface $z = 0$ is unstable.
(c) If $0 > \Sigma^{(2)} > \Sigma^{(1)}$, then we have indeed admissible perturbations with $\omega > 0$ (and $\delta = 0$), and the steady-state solution (3.1) with the interface $z = 0$ is unstable.

The above results assert that the necessary and sufficient condition for the morphological stability of a propagating interphase boundary is $\Sigma^{(1)} > 0$, which implies that $\Sigma^{(2)} > 0$ and therefore may be expressed as

$$V_0^2 < (\Sigma_0/4c\rho) \tag{73}$$

Therefore, the stability condition for a propagating interphase boundary is more restrictive than the thermoplastic stability condition $\Sigma_0 > 0$ (cf. Bai [3]). According to condition (73) the propagating planar interphase boundary will become morphologically unstable when V_0^2 reaches the critical value $(\Sigma_0/4c\rho)$ in at least one of the two phases.

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