Journal of Thermal Stresses

A THEORY OF THERMOVISCOELASTIC DIELECTRICS

Yu-Ning Huang [a] & R. C. Batra [a]

[a] Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, USA
Version of record first published: 27 Apr 2007.

PLEASE SCROLL DOWN FOR ARTICLE

A THEORY OF THERMOVISCOELASTIC DIELECTRICS

Yu-Ning Huang and R. C. Batra

Department of Engineering Science and Mechanics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia, USA

We study finite deformations of heat conducting viscous dielectric solids and derive constitutive relations that satisfy an entropy inequality. These are simplified for orthotropic, transversely isotropic, and isotropic materials. For each class of materials, linear constitutive relations valid for infinitesimal deformations are derived.

Bell [1] documented that the response of all materials is, in general, nonlinear. For piezoelectric ceramics, Lazarus and Crawley [2, 3] stated that the material parameters should be taken to depend upon the induced strains. Norwood et al. [3] and Kulkarni and Hanagud [4] used a Neo-Hookean constitutive relation to model the response of piezoelectric ceramics; the Neo-Hookean model is the simplest model for a nonlinear elastic material. A finite deformation theory of elastic dielectrics was developed, amongst others, by Toupin [5] and Eringen and Maugin [6]. Tiersten [7] used the theory of invariants [8] to derive electroelastic equations that are cubic in the mechanical displacement gradient and the electric potential gradient.

Transversely isotropic piezoelectric materials are widely used in transducers and smart structures. In the latter application it is desirable that these materials exhibit damping, and accordingly these are now being manufactured from composites that show viscous effects. Our goal here is to develop a finite deformation theory for nonmagnetizable but heat-conducting unconstrained dielectrics and to use the theory of invariants to derive the simplest form of constitutive relations for orthotropic, transversely isotropic, and isotropic dielectrics.

BALANCE LAWS

We consider a nonmagnetizable dielectric solid with an infinitesimal memory of mechanical deformations roughly equivalent to that of a Navier–Stokes fluid; the material studied here is simple in the sense of Noll (cf. Truesdell and Noll [9]), is

heat conducting, and exhibits piezoelectric and viscous effects. In spatial description and with quasi-electrostatic approximation, the balance of mass, linear momentum, moment of momentum, internal energy, and Maxwell equations are (Eringen and Maugin [6])

$$
\begin{aligned}
&\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0 \\
&\rho \dot{\mathbf{v}} = \operatorname{div} \hat{\mathbf{t}} + (q_e - \operatorname{div} \mathbf{p}) \mathbf{e} + \rho \mathbf{b} \\
&\mathbf{t}_a = -(\mathbf{p} \otimes \mathbf{e})_a \\
&\rho \dot{\phi} = \operatorname{tr}(\mathbf{tL}) + \operatorname{div} \mathbf{q} - \mathbf{p} \cdot \dot{\mathbf{e}} - \rho h \\
&\operatorname{div} \mathbf{d} = q_e \\
&\operatorname{curl} \mathbf{e} = \mathbf{0}
\end{aligned}
\tag{1}
$$

Here ρ is the present mass density of a material particle, $\mathbf{v}$ is its velocity, $\mathbf{d}$ is the dielectric displacement vector, $\mathbf{p}$ is the polarization vector, $\mathbf{e}$ is the electric field, $\mathbf{b}$ is the body force per unit mass, q_e is the prescribed body charge, $\mathbf{t}$ is the total Cauchy stress given by

$$
\mathbf{t} = \hat{\mathbf{t}} - \mathbf{p} \otimes \mathbf{e}
\tag{2a}
$$

where $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product between vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{t}_a$ is the antisymmetric part of $\mathbf{t}$, and

$$
(\mathbf{a} \otimes \mathbf{b})_a = \mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a}
\tag{2b}
$$

Note that $\hat{\mathbf{t}}$ is symmetric but $\mathbf{t}$ is not. In Eq. $(1)_4$, ϕ is the modified specific internal energy, $\mathbf{L} = \operatorname{grad} \mathbf{v}$ is the velocity gradient, $\mathbf{q}$ is the heat flux per unit present area, and h is the source of internal energy. Vectors $\mathbf{p}$, $\mathbf{e}$, and $\mathbf{d}$ are related by

$$
\mathbf{d} = \mathbf{e} + \mathbf{p}
\tag{3}
$$

The balance laws (1) are supplemented by an entropy inequality, which we take as the Clausius–Duhem inequality (cf. Truesdell and Noll [9]):

$$
\rho \gamma \equiv \rho \dot{\eta} - \operatorname{div}(\mathbf{q}/\theta) - \frac{\rho h}{\theta} \geq 0
\tag{4}
$$

Here η is the specific entropy, $\theta > 0$ is the absolute temperature of a material point, and γ is the specific production of entropy. In terms of the free energy density

$$
\psi = \phi - \eta \theta
\tag{5}
$$

we obtain from Eqs. $(1)_4$ and (4)

$$
-\rho \left(\dot{\psi} + \eta \dot{\theta} \right) + \operatorname{tr}(\mathbf{tL}) - \frac{1}{\theta} \mathbf{q} \cdot \operatorname{grad} \theta - \mathbf{p} \cdot \dot{\mathbf{e}} \geq 0
\tag{6}
$$

as the reduced entropy inequality; here we have eliminated h from $(1)_4$ and (4).

Whereas balance laws (1) are valid for all materials and processes, only those processes that satisfy the entropy inequality (6) are admissible in a real material. Here we adopt the viewpoint that the balance of moment of momentum $(1)_3$ is identically satisfied and the entropy inequality (6) imposes restrictions on the admissible constitutive relations.

CONSTITUTIVE RELATIONS

Anisotropic Materials

Constitutive relations distinguish one material from another. Here we consider a class of solids that exhibit very short memory of the mechanical deformations essentially equivalent to that of Navier–Stokes fluids. For solids the material response also depends upon the choice of the reference configuration, which henceforth is kept fixed. We use the referential description of motion and assume that the response of a material point X is determined by values of

$$\Gamma = \{S, \dot{S}, \theta, G, E\} \tag{7}$$

where

$$2S = F^T F - 1 \qquad G = \partial\theta/\partial X$$
$$E = F^{-1}e \qquad F = \partial x/\partial X \tag{8}$$

$x = x(X, t)$ gives the present position of the material particle that occupied place X in the reference configuration. Let

$$\Lambda = \{\hat{T}, \eta, P, Q, \psi\} \tag{9}$$

where

$$\hat{T} = J F^{-1} \hat{t}(F^{-1})^T \qquad Q = J F^{-1} q \qquad P = J F^{-1} p \qquad J = \det |F| \tag{10}$$

Then our constitutive hypothesis is that

$$\Lambda(X, t) = \mathscr{F}(\Gamma(X, t), X) \tag{11}$$

Henceforth we restrict ourselves to one material point and, therefore, omit the explicit dependence of $\mathscr{F}$ upon X.

Under a change of frame of reference both Γ and Λ are scalar quantities; thus Eqs. (11) are trivially frame-indifferent.

In the terms of Γ and Λ, the reduced entropy inequality (6) becomes

$$-\rho_0\left(\dot{\psi} + \eta\dot{\theta}\right) + \text{tr}(\hat{T}\dot{S}) - \frac{1}{\theta}Q \cdot G - P \cdot \dot{E} \geq 0 \tag{12}$$

With $\rho_0\psi(\mathbf{X}, t) = \Sigma(\Gamma(\mathbf{X}, t))$ and the following usual arguments (see, for example, Coleman and Noll [10]), we obtain

$$\eta = -\frac{1}{\rho_0}\frac{\partial\Sigma}{\partial\theta} \qquad \frac{\partial\Sigma}{\partial\dot{\mathbf{S}}} = \mathbf{0} \qquad \mathbf{P} = -\frac{\partial\Sigma}{\partial\mathbf{E}} \qquad \frac{\partial\Sigma}{\partial\mathbf{G}} = \mathbf{0} \qquad \hat{\mathbf{T}}^e = \frac{\partial\Sigma}{\partial\mathbf{S}} \qquad (13)$$

and

$$\mathscr{R} = \text{tr}(\hat{\mathbf{T}}\dot{\mathbf{S}}) - \frac{1}{\theta}\mathbf{Q}\cdot\mathbf{G} \geq 0 \qquad (14)$$

where

$$\Sigma = \Sigma(\hat{\Gamma}) \qquad \hat{\Gamma} = (\mathbf{S}, \theta, \mathbf{E}) \qquad \hat{\mathbf{T}} = \hat{\mathbf{T}}^e + \hat{\mathbf{T}}^{ne} \qquad \hat{\mathbf{T}}^{ne} = \hat{\mathbf{T}}^{ne}(\Gamma) \qquad (15)$$

Inequality (14) implies that

$$\hat{\mathbf{T}}^{ne}(\mathbf{S}, \mathbf{0}, \theta, \mathbf{0}, \mathbf{E}) = \mathbf{0} \qquad (16a)$$

$$\mathbf{Q}(\mathbf{S}, \mathbf{0}, \theta, \mathbf{0}, \mathbf{E}) = \mathbf{0}$$

A process in which $\dot{\mathbf{S}} = \mathbf{0}$ and $\mathbf{G} = \mathbf{0}$ is referred to as an equilibrium process. Thus in an equilibrium process, $\hat{\mathbf{T}}^{ne}$ and $\mathbf{Q}$ vanish identically. Also

$$\left.\frac{\partial^2\mathscr{R}}{\partial\dot{\mathbf{S}}\partial\mathbf{G}}\right|_{\dot{\mathbf{S}}=\mathbf{G}=\mathbf{0}} \geq 0 \qquad (16b)$$

in the sense that its determinant and all of its subdeterminants must be nonnegative. Functions $\Sigma(\hat{\Gamma})$, $\hat{\mathbf{T}}^{ne}(\Gamma)$, and $\mathbf{Q}(\Gamma)$ satisfying (14) and (16) characterize an anisotropic nonmagnetic dielectric material with a short memory; constitutive relations for η, $\mathbf{P}$, and $\hat{\mathbf{T}}^e$ are obtained by using (13). Depending upon the material symmetries, one can simplify these further.

Isotropic Materials

For isotropic materials, functions Σ, $\hat{\Gamma}^{ne}$, and $\mathbf{Q}$ must satisfy

$$\Sigma(\hat{\bar{\Gamma}}) = \Sigma(\hat{\Gamma}) \qquad \hat{\mathbf{T}}^{ne}(\bar{\Gamma}) = \mathbf{H}\hat{\mathbf{T}}^{ne}(\mathbf{T})\mathbf{H}^T \qquad \mathbf{Q}(\bar{\Gamma}) = \mathbf{H}\mathbf{Q}(\Gamma) \qquad (17)$$

for every orthogonal tensor $\mathbf{H}$ where

$$\hat{\bar{\Gamma}} = \{\mathbf{H}\mathbf{S}\mathbf{H}^T, \theta, \mathbf{H}\mathbf{E}\}$$

$$\bar{\Gamma} = \{\mathbf{H}\mathbf{S}\mathbf{H}^T, \mathbf{H}\dot{\mathbf{S}}\mathbf{H}^T, \theta, \mathbf{H}\mathbf{G}, \mathbf{H}\mathbf{E}\} \qquad (18)$$

Representation theorems (cf. Wang [11], Smith et al. [12], and Zheng [13]) imply that

$$\Sigma = \Sigma(I_1, I_2, \ldots, I_6, \theta) \tag{19}$$

$$\hat{\mathbf{T}}^{ne} = \alpha_0 \mathbf{1} + \alpha_1 \mathbf{S} + \alpha_2 \dot{\mathbf{S}} + \alpha_3 \mathbf{G} \otimes \mathbf{G} + \alpha_4 \mathbf{E} \otimes \mathbf{E}$$

$$+ \alpha_5 (\mathbf{E} \otimes \mathbf{G})_s + \alpha_6 \mathbf{S}^2 + \alpha_7 \dot{\mathbf{S}}^2 + \alpha_8 (\mathbf{G} \otimes \mathbf{SG})_s + \alpha_9 (\mathbf{G} \otimes \mathbf{S}^2 \mathbf{G})_s$$

$$+ \alpha_{10}(\mathbf{E} \otimes \mathbf{SE})_s + \alpha_{11}(\mathbf{E} \otimes \mathbf{S}^2 \mathbf{E})_s + \alpha_{12}(\mathbf{G} \otimes \dot{\mathbf{S}}\mathbf{G})_s + \alpha_{13}(\mathbf{G} \otimes \dot{\mathbf{S}}^2 \mathbf{G})_s$$

$$+ \alpha_{14}(\mathbf{E} \otimes \dot{\mathbf{S}}\mathbf{E})_s + \alpha_{15}(\mathbf{E} \otimes \dot{\mathbf{S}}^2 \mathbf{E})_s + \alpha_{16}(\mathbf{S}\dot{\mathbf{S}})_s + \alpha_{17}(\mathbf{S}^2 \dot{\mathbf{S}})_s + \alpha_{18}(\mathbf{S}\dot{\mathbf{S}}^2)_s$$

$$+ \alpha_{19}(\mathbf{S}(\mathbf{G} \otimes \mathbf{E})_a)_a + \alpha_{20}(\dot{\mathbf{S}}(\mathbf{G} \otimes \mathbf{E})_a)_a \tag{20a}$$

$$\mathbf{Q} = \beta_1 \mathbf{G} + \beta_2 \mathbf{E} + \beta_3 \mathbf{SG} + \beta_4 \mathbf{SE} + \beta_5 \dot{\mathbf{S}}\mathbf{G} + \beta_6 \dot{\mathbf{S}}\mathbf{E}$$

$$+ \beta_7 \mathbf{S}^2 \mathbf{G} + \beta_8 \mathbf{S}^2 \mathbf{E} + \beta_9 \dot{\mathbf{S}}^2 \mathbf{G} + \beta_{10}\dot{\mathbf{S}}^2 \mathbf{E} + \beta_{11}(\mathbf{S}\dot{\mathbf{S}})_a \mathbf{G} + \beta_{12}(\mathbf{S}\dot{\mathbf{S}})_a \mathbf{E} \tag{20b}$$

where

$$(\mathbf{u} \otimes \mathbf{v})_s := \mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u} \qquad (\mathbf{u} \otimes \mathbf{v})_a = \mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u}$$

$$(\mathbf{AB})_a := \mathbf{AB} - \mathbf{BA} \qquad (\mathbf{AB})_s = (\mathbf{AB} + \mathbf{BA})$$

$$I_1 := \mathrm{tr}\,\mathbf{S} \qquad I_2 = \mathrm{tr}\,\mathbf{S}^2 \qquad I_3 = \mathrm{tr}\,\mathbf{S}^3 \qquad I_4 = \mathbf{E} \cdot \mathbf{E} \tag{21}$$

$$I_5 := \mathbf{E} \cdot \mathbf{SE} \qquad I_6 = \mathbf{E} \cdot \mathbf{S}^2 \mathbf{E}$$

Coefficients α_0, $\alpha_1, \ldots, \alpha_{20}$ and $\beta_1, \ldots, \beta_{12}$ are functions of θ, and invariants I_1, $I_2, \ldots, I_{28}$, where

$$I_7 = \mathbf{G} \cdot \mathbf{G} \qquad I_8 = \mathbf{G} \cdot \mathbf{E} \qquad I_9 = \mathrm{tr}\,\dot{\mathbf{S}} \qquad I_{10} = \mathrm{tr}\,\dot{\mathbf{S}}^2 \qquad I_{11} = \mathrm{tr}\,\dot{\mathbf{S}}^3$$

$$I_{12} = \mathbf{G} \cdot \dot{\mathbf{S}}\mathbf{G} \qquad I_{13} = \mathbf{E} \cdot \dot{\mathbf{S}}\mathbf{E} \qquad I_{14} = \mathbf{G} \cdot \mathbf{SG}$$

$$I_{15} = \mathbf{G} \cdot \mathbf{SE} \qquad I_{16} = \mathbf{G} \cdot \dot{\mathbf{S}}\mathbf{E} \qquad I_{17} = \mathbf{G} \cdot \mathbf{S}^2 \mathbf{G}$$

$$\tag{22}$$

$$I_{18} = \mathbf{G} \cdot \dot{\mathbf{S}}^2 \mathbf{G} \qquad I_{19} = \mathbf{E} \cdot \dot{\mathbf{S}}^2 \mathbf{E} \qquad I_{20} = \mathbf{G} \cdot \mathbf{S}\dot{\mathbf{S}}\mathbf{G} \qquad I_{21} = \mathbf{E} \cdot \mathbf{S}\dot{\mathbf{S}}\mathbf{E}$$

$$I_{22} = \mathbf{G} \cdot \mathbf{S}^2 \mathbf{E} \qquad I_{23} = \mathbf{G} \cdot \dot{\mathbf{S}}^2 \mathbf{E} \qquad I_{24} = \mathbf{G} \cdot (\dot{\mathbf{S}}\mathbf{S})_a \mathbf{G}$$

$$I_{25} = \mathrm{tr}(\mathbf{S}\dot{\mathbf{S}}) \qquad I_{26} = \mathrm{tr}(\mathbf{S}^2 \dot{\mathbf{S}}) \qquad I_{27} = \mathrm{tr}(\mathbf{S}\dot{\mathbf{S}}^2) \qquad \mathbf{I}_{28} = \mathrm{tr}(\mathbf{S}^2 \dot{\mathbf{S}}^2)$$

In order that Eqs. (20a, b) satisfy (16a)

$$\bar{\alpha}_0 = \bar{\alpha}_1 = \bar{\alpha}_4 = \bar{\alpha}_6 = \bar{\alpha}_{10} = \bar{\alpha}_{11} = 0$$

$$\bar{\beta}_2 = \bar{\beta}_4 = \bar{\beta}_8 = 0 \tag{23}$$

where a superimposed bar indicates their values when $\dot{\mathbf{S}} = \mathbf{G} = 0$.
Equations (13) and (19) yield

$$\hat{\mathbf{T}}^e = \gamma_1 \mathbf{1} + 2\gamma_2 \mathbf{S} + 3\gamma_3 \mathbf{S}^2 + \gamma_5 \mathbf{E} \otimes \mathbf{E} + \gamma_6 (\mathbf{SE} \otimes \mathbf{E})_s$$

$$\mathbf{P} = -2[\gamma_4 \mathbf{E} + \gamma_5 \mathbf{SE} + \gamma_6 \mathbf{S}^2 \mathbf{E}] \tag{24}$$

$$\eta = \frac{1}{\rho_0} \frac{\partial \Sigma}{\partial \theta} = \hat{\eta}(I_1, \dots, I_6, \theta)$$

where $\gamma_i = \partial \Sigma / \partial I_i$, $i = 1, 2, \dots, 6$.

The challenging (probably insurmountable) task now is to devise experiments to determine the functional dependence of Σ and $\alpha_1, \alpha_1, \dots, \alpha_{20}$; $\beta_1, \dots, \beta_{12}$ upon their arguments. Since isotropic materials have the largest symmetry group, they require the determination of a minimum number of functions.

Transversely Isotropic Materials

Piezoceramics are usually poled in a certain direction and can be reasonably modeled as transversely isotropic. Let the unit vector $\mathbf{N}$ be the axis of transverse isotropy. The representation theorem of Zheng [13] yields

$$\Sigma = \Sigma(J_1, J_2, \dots, J_{10}, \theta) \tag{25}$$

$$\hat{\mathbf{T}}^{ne} = \alpha_0 \mathbf{1} + \alpha_1 \mathbf{S} + \alpha_2 \dot{\mathbf{S}} + \alpha_3 \mathbf{S}^2 + \alpha_4 \dot{\mathbf{S}}^2 + \alpha_5 (\mathbf{S}\dot{\mathbf{S}})_s$$

$$+ \alpha_6 \mathbf{E} \otimes \mathbf{E} + \alpha_7 \mathbf{G} \otimes \mathbf{G} + \alpha_8 (\mathbf{G} \otimes \mathbf{E})_s + \alpha_9 \mathbf{N} \otimes \mathbf{N}$$

$$+ \alpha_{10} (\mathbf{N} \otimes \mathbf{SN})_s + \alpha_{11} (\mathbf{N} \otimes \mathbf{S}^2 \mathbf{N})_s + \alpha_{12} (\mathbf{N} \otimes \dot{\mathbf{S}} \mathbf{N})_s$$

$$+ \alpha_{13} (\mathbf{N} \otimes \dot{\mathbf{S}}^2 \mathbf{N})_s + \alpha_{14} (\mathbf{N} \otimes \mathbf{E})_s + \alpha_{15} (\mathbf{N} \otimes \mathbf{G})_s$$

$$+ \alpha_{16} (\mathbf{S}(\mathbf{N} \otimes \mathbf{E})_a)_a + \alpha_{17} (\mathbf{S}(\mathbf{N} \otimes \mathbf{G})_a)_a + \alpha_{18} (\dot{\mathbf{S}}(\mathbf{N} \otimes \mathbf{E})_a)_a$$

$$+ \alpha_{19} (\dot{\mathbf{S}}(\mathbf{N} \otimes \mathbf{G})_a)_a \tag{26a}$$

$$\mathbf{Q} = \beta_1 \mathbf{E} + \beta_2 \mathbf{G} + \beta_3 \mathbf{SE} + \beta_4 \mathbf{SG} + \beta_5 \dot{\mathbf{S}} \mathbf{E} + \beta_6 \dot{\mathbf{S}} \mathbf{G} + \beta_7 \mathbf{N}$$

$$+ \beta_8 \mathbf{SN} + \beta_9 \dot{\mathbf{S}} \mathbf{N} + \beta_{10} \mathbf{S}^2 \mathbf{N} + \beta_{11} \dot{\mathbf{S}}^2 \mathbf{N} + \beta_{12} (\mathbf{S}\dot{\mathbf{S}})_a \mathbf{N} \tag{26b}$$

where

$$(S(N \otimes E)_a)_a = S(N \otimes E - E \otimes N) - (N \otimes E - E \otimes N)S \tag{27}$$

$\alpha_0, \ldots, \alpha_{19}, \beta_1, \ldots, \beta_{12}$ are functions of θ and the invariants $J_1, \ldots, J_{33}$. These invariants are defined as

$$J_i = I_i \qquad i = 1, \ldots, 5$$

$$J_i = I_{i-2} \qquad i = 11, \ldots, 18 \qquad J_i = I_{i+6} \qquad i = 19, 20, 21$$

$$J_6 = N \cdot SN \qquad J_{22} = N \cdot \dot{S}N \qquad J_7 = N \cdot S^2 N \qquad J_{23} = N \cdot \dot{S}^2 N$$

$$J_8 = N \cdot E \qquad J_{25} = N \cdot G \qquad J_9 = N \cdot SE \qquad J_{24} = N \cdot S\dot{S}N \tag{28}$$

$$J_{10} = N \cdot S^2 E \qquad J_{27} = N \cdot SG \qquad J_{28} = N \cdot \dot{S}E \qquad J_{29} = N \cdot \dot{S}G$$

$$J_{31} = N \cdot S^2 G \qquad J_{32} = N \cdot \dot{S}^2 E \qquad J_{33} = N \cdot \dot{S}^2 G \qquad J_{26} = N \cdot S\dot{S}E \qquad J_{30} = N \cdot S\dot{S}G$$

For Eqs. (26a, b) to satisfy Eqs. (16a) we must have

$$\bar{\alpha}_0 = \bar{\alpha}_1 = \bar{\alpha}_3 = \bar{\alpha}_6 = \bar{\alpha}_9 = \bar{\alpha}_{10} = \bar{\alpha}_{11} = \bar{\alpha}_{14} = \bar{\alpha}_{16} = 0$$

$$\bar{\beta}_1 = \bar{\beta}_3 = \bar{\beta}_7 = \bar{\beta}_8 = \bar{\beta}_{10} = 0 \tag{29}$$

where a superimposed bar indicates their values when $\dot{S} = G = 0$. Relations $(13)_3$ and $(13)_5$ imply that

$$\hat{T}^e = \gamma_1 1 + 2\gamma_2 S + 3\gamma_3 S^2 + \gamma_5 E \otimes E + \gamma_6 N \otimes N$$

$$+ \gamma_7 (N \otimes SN)_s + \frac{1}{2}\gamma_9 (N \otimes E)_s + \gamma_{10}(N \otimes SE)_s$$

$$P = -\left[2\gamma_4 E + 2\gamma_5 SE + \gamma_8 N + \gamma_9 SN + \gamma_{10} S^2 N\right] \tag{30}$$

$$\eta = -\frac{1}{\rho_0}\partial\Sigma/\partial\theta = \hat{\eta}(J_1, J_2, \ldots, J_{10}, \theta)$$

where $\gamma_i = \partial\Sigma/\partial J_i$.

Orthotropic Materials

An orthotropic material point has three orthogonal planes of symmetry; let e_1, e_2, e_3 denote unit vectors perpendicular to these planes and

$$M = e_2 \otimes e_2 - e_3 \otimes e_3 \tag{31}$$

The representation theorem of Zheng [13] yields

$$\Sigma = \Sigma(I_1, \ldots, I_6, K_1, \ldots, K_7, \theta) \tag{32}$$

$$\hat{\mathbf{T}}^{ne} = \alpha_0 \mathbf{1} + \alpha_1 \mathbf{S} + \alpha_2 \dot{\mathbf{S}} + \alpha_3 \mathbf{S}^2 + \alpha_4 \dot{\mathbf{S}}^2 + \alpha_5 (\mathbf{S}\dot{\mathbf{S}})_s + \alpha_6 \mathbf{E} \otimes \mathbf{E}$$

$$+ \alpha_7 \mathbf{G} \otimes \mathbf{G} + \alpha_8 (\mathbf{E} \otimes \mathbf{SE})_s + \alpha_9 (\mathbf{E} \otimes \dot{\mathbf{S}}\mathbf{E})_s + \alpha_{10} (\mathbf{G} \otimes \mathbf{SG})_s$$

$$+ \alpha_{11} (\mathbf{G} \otimes \dot{\mathbf{S}}\mathbf{G})_s + \alpha_{12} (\mathbf{E} \otimes \mathbf{G})_s + \alpha_{13} (\mathbf{S}(\mathbf{E} \otimes \mathbf{G})_a)_a$$

$$+ \alpha_{14} (\dot{\mathbf{S}}(\mathbf{E} \otimes \mathbf{G})_a)_a + \alpha_{15} \mathbf{M} + \alpha_{16} \mathbf{e}_1 \otimes \mathbf{e}_1$$

$$+ \alpha_{17} (\mathbf{MS})_s + \alpha_{18} (\mathbf{M}\dot{\mathbf{S}})_s + \alpha_{19} (\mathbf{e}_1 \otimes \mathbf{Se}_1)_s + \alpha_{20}(\mathbf{e}_1 \otimes \dot{\mathbf{S}}\mathbf{e}_1)_s$$

$$+ \alpha_{21} (\mathbf{E} \otimes \mathbf{ME})_s + \alpha_{22} (\mathbf{G} \otimes \mathbf{MG})_s + \alpha_{23} (\mathbf{E} \cdot \mathbf{e}_1)(\mathbf{e}_1 \otimes \mathbf{E})_s$$

$$+ \alpha_{24} (\mathbf{G} \cdot \mathbf{e}_1)(\mathbf{e}_1 \otimes \mathbf{G})_s + \alpha_{25} (\mathbf{M}(\mathbf{E} \otimes \mathbf{G})_a)_a \tag{33a}$$

$$\mathbf{Q} = \beta_1 \mathbf{E} + \beta_2 \mathbf{G} + \beta_3 \mathbf{SE} + \beta_4 \mathbf{SG} + \beta_5 \dot{\mathbf{S}}\mathbf{E} + \beta_6 \dot{\mathbf{S}}\mathbf{G} + \beta_7 \mathbf{S}^2 \mathbf{E} + \beta_8 \mathbf{S}^2 \mathbf{G}$$

$$+ \beta_9 \dot{\mathbf{S}}^2 \mathbf{E} + \beta_{10} \dot{\mathbf{S}}^2 \mathbf{G} + \beta_{11} \mathbf{ME} + \beta_{12} \mathbf{MG} + \beta_{13} (\mathbf{E} \cdot \mathbf{e}_1)\mathbf{e}_1 + \beta_{14} (\mathbf{G} \cdot \mathbf{e}_1)\mathbf{e}_1$$

$$+ \beta_{15} (\mathbf{MS})_a \mathbf{E} + \beta_{16} (\mathbf{MS})_a \mathbf{G} + \beta_{17} (\mathbf{M}\dot{\mathbf{S}})_a \mathbf{E} + \beta_{18} (\mathbf{M}\dot{\mathbf{S}})_a \mathbf{G} \tag{33b}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{25}$; $\beta_1, \ldots, \beta_{18}$ are functions of θ and invariants $I_1, \ldots, I_{27}$; $K_1, \ldots, K_{21}$. Invariants $K_1, \ldots, K_{21}$ are defined below.

$$K_1 = \text{tr}(\mathbf{MS}) \qquad K_2 = \text{tr}(\mathbf{MS}^2) \qquad K_3 = \mathbf{e}_1 \cdot \mathbf{Se}_1$$

$$K_4 = \mathbf{e}_1 \cdot \mathbf{S}^2 \mathbf{e}_1 \qquad K_5 = \mathbf{E} \cdot \mathbf{ME} \qquad K_6 = (\mathbf{E} \cdot \mathbf{e}_1)^2 \qquad K_7 = \mathbf{E} \cdot \mathbf{MSE}$$

$$K_8 = \text{tr}(\mathbf{M}\dot{\mathbf{S}}) \qquad K_9 = \text{tr}(\mathbf{M}\dot{\mathbf{S}}^2) \qquad K_{10} = \mathbf{e}_1 \cdot \dot{\mathbf{S}}\mathbf{e}_1 \qquad K_{11} = \mathbf{e}_1 \cdot \dot{\mathbf{S}}^2 \mathbf{e}_1$$

$$K_{12} = \text{tr}(\mathbf{MS}\dot{\mathbf{S}}) \qquad K_{13} = \mathbf{G} \cdot \mathbf{MG} \qquad K_{14} = (\mathbf{G} \cdot \mathbf{e}_1)^2 \tag{34}$$

$$K_{15} = \mathbf{G} \cdot \mathbf{MSG} \qquad K_{16} = \mathbf{E} \cdot \mathbf{M}\dot{\mathbf{S}}\mathbf{E} \qquad K_{17} = \mathbf{G} \cdot \mathbf{M}\dot{\mathbf{S}}\mathbf{G} \qquad K_{18} = E \cdot \mathbf{MG}$$

$$K_{19} = (\mathbf{E} \cdot \mathbf{e}_1)(\mathbf{G} \cdot \mathbf{e}_1) \qquad K_{20} = \mathbf{E} \cdot (\mathbf{MS})_a \mathbf{G} \qquad K_{21} = \mathbf{E} \cdot (\mathbf{M}\dot{\mathbf{S}})_a \mathbf{G}$$

Equations (33a, b) and (16) imply that

$$\bar{\alpha}_0 = \bar{\alpha}_1 = \bar{\alpha}_3 = \bar{\alpha}_6 = \bar{\alpha}_8 = \bar{\alpha}_{15} = \bar{\alpha}_{16} = \bar{\alpha}_{17} = \bar{\alpha}_{19} = \bar{\alpha}_{21} = \bar{\alpha}_{23} = 0$$

$$\bar{\beta}_1 = \bar{\beta}_3 = \bar{\beta}_7 = \bar{\beta}_{11} = \bar{\beta}_{13} = \bar{\beta}_{15} = 0 \tag{35}$$

Substitution from Eq. (32) into Eqs. $(13)_3$ and $(13)_5$ yields

$$\mathbf{P} = -2\big[\gamma_4\mathbf{E} + \gamma_5\mathbf{SE} + \gamma_6\mathbf{S}^2\mathbf{E} + \delta_5\mathbf{ME} + \delta_6(\mathbf{E}\cdot\mathbf{e}_1)\mathbf{e}_1 + \delta_7\mathbf{MSE}\big] \qquad (36)$$

$$
\begin{aligned}
\hat{\mathbf{T}}^e = {} & \gamma_1\mathbf{1} + 2\gamma_2\mathbf{S} + 3\gamma_3\mathbf{S}^2 + \gamma_5\mathbf{E}\otimes\mathbf{E} + \gamma_6(\mathbf{SE}\otimes\mathbf{E})_s \\
& + \delta_1\mathbf{M} + \delta_2(\mathbf{MS})_s + \delta_3\mathbf{e}_1\otimes\mathbf{e}_1 + 2\delta_4(\mathbf{e}_1\otimes\mathbf{Se}_1)_s \\
& + \frac{1}{2}\delta_7(\mathbf{E}\otimes\mathbf{ME})_s
\end{aligned}
\qquad (37)
$$

$$\eta = -\frac{1}{\rho_0}\partial\Sigma/\partial\theta = \hat{\eta}(I_1,\ldots,I_6, K_1,\ldots,K_7,\theta)$$

where $\gamma_i = \dfrac{\partial\Sigma}{\partial I_i}$, $\delta_i = \dfrac{\partial\Sigma}{\partial K_i}$.

LINEAR CONSTITUTIVE RELATIONS

Linear constitutive relations are appropriate when the magnitudes of displacement gradients and, therefore, of the infinitesimal strain tensor $\boldsymbol{\epsilon}$, its time rate of change $\dot{\boldsymbol{\epsilon}}$, electric field $\mathbf{E}$, the relative temperature change $\hat{\theta} = (\theta - \theta_0)/\theta_0$, and of the gradients of the temperature change are small as compared to one. Since linear theories are easy to work with, we give below linear constitutive relations for different material symmetries.

Anisotropic Materials

We assume that Σ, $\hat{\mathbf{T}}^{ne}$, and $\mathbf{Q}$ are smooth functions of their arguments and can be expanded in terms of a Taylor series about the values their arguments take in the reference configuration. The finite strain tensor $\mathbf{S}$ is expressed in terms of displacement gradients. Using Eqs. $(13)_{1,3,5}$; keeping terms of order one in $\boldsymbol{\epsilon}$, $\dot{\boldsymbol{\epsilon}}$, $\hat{\theta}$, $\hat{\mathbf{G}} \equiv \mathrm{Grad}\,\hat{\theta}$, and $\mathbf{E}$; and recalling Eq. (16), we arrive at the following linear constitutive relations:

$$\hat{\mathbf{T}}^e = \hat{\mathbf{T}}_0^e + \boldsymbol{\alpha}^e\hat{\theta} + \mathbf{C}^e\boldsymbol{\epsilon} - \mathbf{D}^e\mathbf{E}$$

$$\hat{\mathbf{T}}^{ne} = \mathbf{C}^{ne}\dot{\boldsymbol{\epsilon}} + \boldsymbol{\alpha}^{ne}\hat{\mathbf{G}}$$

$$\mathbf{P} = \mathbf{P}_0 + \boldsymbol{\omega}\hat{\theta} + \mathbf{AE} + \mathbf{D}^e\boldsymbol{\epsilon} \qquad (38)$$

$$\mathbf{Q} = -\mathbf{K}\hat{\mathbf{G}} + \mathbf{K}^1\dot{\boldsymbol{\epsilon}}$$

$$\eta = \eta_0 + \frac{1}{\rho_0}\big(\mu\hat{\theta} + \boldsymbol{\mu}^1\cdot\boldsymbol{\epsilon} + \boldsymbol{\omega}\cdot\mathbf{E}\big)$$

Here $\hat{\mathbf{T}}_0^e$, $\mathbf{P}_0$, and η_0 are the values of the equilibrium stress $\mathbf{T}^e$, polarization $\mathbf{P}$, and the specific entropy η in the reference configuration. Necessary conditions for Eqs. $(38)_{2,4}$ to satisfy the reduced entropy inequality (14) imply that the fourth-order viscosity tensor $\mathbf{C}^{ne}$ and the second-order thermal conductivity tensor $\mathbf{K}$ are positive definite; however, K need not be symmetric. The fourth-order tensor $\mathbf{C}^e$ is called the elasticity tensor and the usual symmetry $\mathbf{C}^e = \mathbf{C}^{eT}$ assumed for it is an additional assumption rather than a consequence of the entropy inequality, components of the second-order tensor $\boldsymbol{\alpha}^e$ are the thermal stress moduli, of the third-order tensor $\mathbf{D}^e$ are the piezoelectric moduli, $\boldsymbol{\omega}$ may be called the pyroelectric polarizability vector, and the second-order tensor $\mathbf{A}$ is the dielectric susceptibility tensor. Tensors such as $\boldsymbol{\alpha}^e$ and $\mathbf{C}^e$ are functions of $\mathbf{X}$ only and are constants for a homogeneous material.

Isotropic Materials

For isotropic materials, Eqs. (38) simplify to

$$\hat{\mathbf{T}}^e = t_0 \mathbf{1} + \alpha\hat{\theta}\mathbf{1} + \lambda_0(\text{tr }\boldsymbol{\epsilon})\mathbf{1} + 2\mu_0\boldsymbol{\epsilon}$$

$$\hat{\mathbf{T}}^{ne} = \lambda_1(\text{tr }\dot{\boldsymbol{\epsilon}})\mathbf{1} + 2\mu_1\dot{\boldsymbol{\epsilon}}$$

$$\mathbf{P} = a\mathbf{E} \tag{39}$$

$$\mathbf{Q} = -k\hat{\mathbf{G}}$$

$$\eta = \eta_0 + \alpha \text{ tr }\boldsymbol{\epsilon} + \beta\hat{\theta}$$

where λ_0 and μ_0 are the Lamé's constants, α is the coefficient of thermal expansion, λ_1 and μ_1 are the viscosities of the material, a is the dielectric constant, and $k > 0$ is the thermal conductivity of the material. As expected, these materials do not exhibit any piezoelectric effect.

Transversely Isotropic Materials

Linear constitutive relations for transversely isotropic materials derived by using the procedure outlined above are given as

$$\hat{\mathbf{T}}^e = \left(\gamma_0 + \gamma_1\hat{\theta} + \gamma_2\mathbf{N}\cdot\boldsymbol{\epsilon}\mathbf{N} + \gamma_3\mathbf{N}\cdot\mathbf{E} + 2\gamma_4(\text{tr }\boldsymbol{\epsilon})\right)\mathbf{1}$$

$$+ \left(\gamma_5 + \gamma_2(\text{tr }\boldsymbol{\epsilon})\right.$$

$$\left. + 2\gamma_6\mathbf{N}\cdot\boldsymbol{\epsilon}\mathbf{N} + \gamma_7\mathbf{N}\cdot\mathbf{E} + \gamma_8\hat{\theta}\right)\mathbf{N}\otimes\mathbf{N} + \gamma_9\boldsymbol{\epsilon}$$

$$+ \gamma_{10}(\mathbf{N}\otimes\boldsymbol{\epsilon}\mathbf{N})_s + \gamma_{11}(\mathbf{N}\otimes\mathbf{E})_s \tag{40}$$

$$\hat{\mathbf{T}}^{ne} = \alpha_1 \dot{\boldsymbol{\epsilon}} + \alpha_2 (\mathbf{N} \otimes \dot{\boldsymbol{\epsilon}} \mathbf{N})_s + \alpha_4 (\mathbf{N} \otimes \hat{\mathbf{G}})_s$$

$$\mathbf{Q} = -\beta_2 \hat{\mathbf{G}} + \beta_3 \dot{\boldsymbol{\epsilon}} \mathbf{N}$$

$$\mathbf{P} = \gamma_{12} \mathbf{E} + \left(\gamma_{13} + \gamma_3 (\mathrm{tr}\,\boldsymbol{\epsilon}) + \gamma_7 \mathbf{N} \cdot \boldsymbol{\epsilon} \mathbf{N} + \gamma_{13} \mathbf{N} \cdot \mathbf{E} + \gamma_{14} \hat{\theta} \right) \mathbf{N} + \gamma_{11} \boldsymbol{\epsilon} \mathbf{N}$$

$$\eta = \eta_0 - \frac{1}{\rho_0} \left(\gamma_1 \,\mathrm{tr}\,\boldsymbol{\epsilon} + \gamma_8 \mathbf{N} \cdot \boldsymbol{\epsilon} \mathbf{N} + \gamma_{15} \hat{\theta} + \gamma_{14} \mathbf{N} \cdot \mathbf{E} \right)$$

(40)
(*Cont.*)

Recall that $\mathbf{N}$ is a unit vector in the direction of transverse isotropy.

Orthotropic Materials

Linear constitutive relations for an orthotropic material with planes of symmetries perpendicular to orthonormal vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are

$$\hat{\mathbf{T}}^e = \left(\alpha_1 + \alpha_{11}\,\mathrm{tr}(\mathbf{M}\boldsymbol{\epsilon}) + \alpha_{12}\mathbf{e}_1 \cdot \boldsymbol{\epsilon}\mathbf{e}_1 + \alpha_{13}\hat{\theta} + 2\alpha_{16}\,\mathrm{tr}\,\boldsymbol{\epsilon} \right)\mathbf{1}$$

$$+ 2\alpha_2\,\boldsymbol{\epsilon} + \left(\alpha_4 + \alpha_{11}\,\mathrm{tr}\,\boldsymbol{\epsilon} + 2\alpha_{17}\,\mathrm{tr}(\mathbf{M}\boldsymbol{\epsilon}) + \alpha_{18}\mathbf{e}_1 \cdot \boldsymbol{\epsilon}\mathbf{e}_1 + \alpha_{19}\hat{\theta} \right)\mathbf{M}$$

$$+ \left(\alpha_6 + \alpha_{12}\,\mathrm{tr}\,\boldsymbol{\epsilon} + \alpha_{18}\,\mathrm{tr}(\mathbf{M}\boldsymbol{\epsilon}) + 2\alpha_{20}(\mathbf{e}_1 \cdot \boldsymbol{\epsilon}\mathbf{e}_1) + \alpha_{21}\hat{\theta} \right)\mathbf{e}_1 \otimes \mathbf{e}_1$$

$$+ 2\alpha_5 (\mathbf{M} \otimes \boldsymbol{\epsilon})_s + \alpha_7 \mathbf{e}_1 \otimes \boldsymbol{\epsilon}\mathbf{e}_1$$

$$\hat{\mathbf{T}}^{ne} = \gamma_9 \dot{\boldsymbol{\epsilon}} + \gamma_{10}(\mathbf{M}\dot{\boldsymbol{\epsilon}})_s + \gamma_{11}(\mathbf{e}_1 \otimes \dot{\boldsymbol{\epsilon}}\mathbf{e}_1)_s$$

$$\mathbf{P} = 2(\alpha_3 \mathbf{E} + \alpha_8 \mathbf{M}\mathbf{E} + \alpha_9 (\mathbf{E} \cdot \mathbf{e}_1)\mathbf{e}_1)$$

$$\mathbf{Q} = -\beta_1 \hat{\mathbf{G}} + \beta_2 \mathbf{M}\hat{\mathbf{G}} + \beta_3 (\mathbf{e}_1 \cdot \hat{\mathbf{G}})\mathbf{e}_1$$

$$\eta = \eta_0 - \frac{1}{\rho_0} \left(\alpha_{10} + \alpha_{13}\,\mathrm{tr}\,\boldsymbol{\epsilon} + \alpha_{19}\,\mathrm{tr}(\mathbf{M}\boldsymbol{\epsilon}) + \alpha_{21}(\mathbf{e}_1 \cdot \boldsymbol{\epsilon}\mathbf{e}_1) + 2\alpha_{23}\hat{\theta} \right)$$

(41)

CONCLUSIONS

We derived constitutive relations for finite deformations of orthotropic, transversely isotropic, and isotropic dielectrics and deduced from them constitutive relations appropriate for infinitesimal deformations of these materials. From the constitutive relations for finite deformations, one could also derive those for small mechanical deformations, infintesimal temperature changes, and temperature gradients but strong electric fields and also a second-order theory in which terms up to second order in displacement gradients, temperature gradients, relative temperature changes, and electric fields are kept; see, for example, Yang and Batra [14, 15].

REFERENCES

1. J. F. Bell, The Experimental Foundations of Solid Mechanics, in C. Truesdell (ed.), *Handbuch der Physik*, vol. via/1, pp. 1–813, Springer, Berlin, Heidelberg, and New York, 1973.
2. K. B. Lazarus and E. F. Crawley, *Induced Strain Actuation of Composite Plates*, GLT Report 197, MIT, 1989.
3. D. S. Norwood, M. J. Shuart, and C. T. Herakovich, Geometrically Nonlinear Analysis of Interlaminar Stresses in Unsymmetrically Laminated Plates Subject to Inplane Mechanical Loading, in *Proc. AIAA/ASME/ASCE/AHS/ACE 32nd Structures, Structural Dynamics and Materials Conference*, pp. 938–955, 1991.
4. G. Kulkarni and S. Hanagud, Modeling Issues in the Vibration Control with Piezoceramic Actuators, in *Smart Structures and Materials*, ASME AD-Vol. 24/AMD-Vol. 123, pp. 7–17, ASME Press, New York, 1991.
5. R. A. Toupin, The Elastic Dielectric, *J. Rational Mech. Anal.*, vol. 5, pp. 849–915, 1956.
6. A. C. Eringen and G. A. Maugin, *Electrodynamics of Continua*. I, Springer, Berlin, Heidelberg, and New York, 1990.
7. H. F. Tiersten, Nonlinear Electroelastic Equations Cubic in the Small Field Variables, *J. Acoustic Soc. Amer.*, vol. 57, pp. 660–666, 1975.
8. A. J. M. Spencer, *Theory of Invariants in Continuum Physics*. 1 (A. C. Eringen, ed.), pp. 292–307, Academic Press, New York, 1971.
9. C. Truesdell and W. Noll, The Non-Linear Field Theories of Mechanics, in S. Flügge (ed.), *Handbuch der Physik*, vol. III, Springer, Berlin, Heidelberg, and New York, 1965.
10. B. D. Coleman and W. Noll, The Thermodynamics of Elastic Materials with Heat Conduction and Viscosity, *Arch. Rational Mech. Anal.*, vol. 13, pp. 167–178, 1963.
11. C. C. Wang, A General Representation Theorem for Constitutive Relations, *Arch. Rational Mech. Anal.*, vol. 32, pp. 1–25, 1969.
12. G. F. Smith, M. M. Smith, and R. S. Rivlin, Integrity Bases for a Symmetric Tensor and a Vector—the Crystal Classes, *Arch. Rational Mech. Anal.*, vol. 12, pp. 93–133, 1963.
13. Q.-S. Zheng, On Transversely Isotropic, Orthotropic and Relatively Isotropic Functions of Symmetric Tensors, Skew-Symmetric Tensors, and Vectors, *Int. J. Engrg. Sci.*, vol. 31, pp. 1399–1453, 1993.
14. J. S. Yang and R. C. Batra, A Second-Order Theory for Piezoelectric Materials, *J. Acous. Soc. Amer.*, vol. 97, pp. 280–288, 1995.
15. R. C. Batra and J. S. Yang, Second-order Constitutive Relations for Transversely Isotropic Piezoelectric Porous Materials, *J. Acous. Soc. Amer.*, vol. 97, pp. 2595–2598, 1995.