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On the Motion of an Elastic Body Rolling/Sliding on an Elastic Substrate

The three-dimensional evolutionary problem of rolling/sliding of a linear elastic body on a linear elastic substrate is studied. The inertial properties of the body regarded as rigid are accounted for. By employing an asymptotic analysis, it is shown that the process can be divided into two phases: transient and quasistationary. An expression for the frictional force as a function of the externally applied forces and moments, and inertial properties of the body is derived. For an ellipsoid rolling/sliding on a linear elastic substrate, numerical results for the frictional force distribution, slip/adhesion subareas, and the evolution of the slip velocity are given.

1 Introduction

The basic stationary model of frictional contact interaction (e.g., see Johnson, 1985 and Kalker, 1990) considers such important effects as the local slip and adhesion subareas, infinitesimal elastic deformations of contacting bodies and their rigid body kinematics. It aids in the simulation of friction, wear, and fatigue in rolling bearings, gears, railway and crane wheels, and other mechanical and structural elements. Kalker (1970) has examined rolling/sliding contact problems wherein transient effects have been important. He studied the contact between two identical cylinders when the resultant frictional force between them is independent of time, and showed that the transient effects lasted for the time interval required for the center of a cylinder to travel a distance equal to several widths of the contact area. This analysis did not account for the inertial properties of the contacting bodies. Since boundary conditions at the contacting surfaces involve rigid body kinematics, it is imperative that a realistic model of the interaction between moving solids include the rigid body dynamics.

Here we study the interaction between the frictional forces at the contact surface and the rigid-body dynamics of a linear elastic body rolling/sliding on an elastic foundation. Both the body and the substrate may be covered by linear elastic deformable layers. We assume that none of these bodies exhibits creep, and show that there exists a boundary layer effect and estimate the duration of the initial transient phase during which this boundary layer effect is important. Subsequently, the problem can be considered as quasistationary. It is shown that the expression for the frictional force involves coefficients that depend strongly upon the distribution of mass in the body.

We note that problems of the type considered herein have been studied by vehicle system dynamicists (e.g., see the proceedings edited by Mariagileti, 1993 and Kortum and Sharp, 1993). However, they are primarily interested in the motion of rail vehicles and we focus on studying the mechanics of contacting surfaces.

For a linear elastic body in the form of an ellipsoid rolling/sliding on a linear elastic substrate, we give numerical results for the distribution of the frictional force, slip/adhesion subareas, and the evolution of the slip velocity. We illustrate the effects of inertia forces and the two-dimensional character of the distribution of frictional forces.

We should add that Kikuchi and Oden (1988), Komvopoulos (1989), and Bhargava et al. (1985) have considered some aspects of nonlinearities, viscosity, and plastic deformations of the contacting bodies; the latter two works did not consider the effect of frictional forces at the contacting surface, and the main focus of Kikuchi and Oden's work was the solution of the problem by the finite element method.

2 Formulation of the Problem

We consider the motion of a body of revolution that is abutting the flat surface of an elastic substrate. Both the body and the substrate may be covered by thin deformable layers whose deformations at the contact surface are described by local functions of contact stresses. The kinematics of the body is determined by two components of the velocity of its center of mass, and two components of the angular velocity of the body regarded as absolutely rigid (e.g., see Fig. 1). This model

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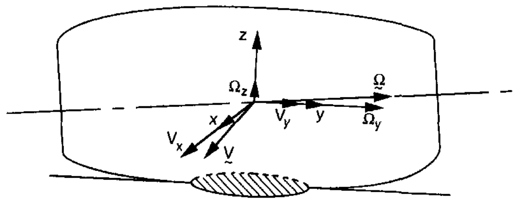

Fig. 1 Rolling/sliding of a body on a substrate

includes the typical kinematics of the railway and crane wheels, and the ball, cylindrical and conical rolling bearings. Points within the contact area have longitudinal and lateral components of the slip velocity. We assume that the body and the substrate have originally a point contact, and the regime of rolling/sliding interaction, defined by

$$|\mathbf{s}| << V_c \tag{1}$$

is realized. Here V_c is the typical speed of the center of mass of the body. The slip velocity $\mathbf{s}$ is given by

$$\mathbf{s} = \mathbf{v} + \frac{d}{dt}(\mathbf{u}^b - \mathbf{u}^s) = \mathbf{v} - V_x \frac{\partial}{\partial x}(\mathbf{u}^b - \mathbf{u}^s) + \frac{\partial}{\partial t}(\mathbf{u}^b - \mathbf{u}^s) \tag{2}$$

The slip velocity $\mathbf{v}$ of the body regarded as absolutely rigid is found from

$$v_x = V_x - \Omega_y R_x - \Omega_z y, \quad v_y = V_y + \Omega_z x. \tag{3}$$

The boundary conditions at the contact surfaces are

$$|\tau| \le \mu p \text{ if } |\mathbf{s}| = 0, \tag{4}$$

$$|\tau| = \mu p \frac{\mathbf{s}}{|\mathbf{s}|} \text{ if } |\mathbf{s}| > 0, \tag{5}$$

and describe, respectively, the adhesion and slip subareas. We will analyze the evolution of frictional forces and slip velocities in a later section. We neglect the effect of frictional forces on the pressure distribution which is generally very weak for most practical cases (see e.g., Johnson, 1985). The problem of the determination of the pressure on the contact surface is not considered here; in many engineering problems involving local contact interaction, the Hertzian solution of a parabolic distribution can be used.

As was also done by Kalker (1970) we assume that the body and the substrate can be approximated as half spaces within the vicinity of the contact area. Thus the tangential displacements of the body and the substrate are determined by the Boussinesq-Cerruti's formulae (e.g., see Love, 1944) and local terms, viz.,

$$\mathbf{u}^b = \mathbf{B}^b(p) + \mathbf{B}^b(\tau) - k_b\tau, \tag{6}$$

$$\mathbf{u}^s = \mathbf{B}^s(p) + \mathbf{B}^s(\tau) + k_s\tau. \tag{7}$$

Explicit expressions for $\mathbf{B}^b(p)$, $\mathbf{B}^b(\tau)$, $\mathbf{B}^s(p)$, and $\mathbf{B}^s(\tau)$ are given in the Appendix. We use dynamic equations

$$M\dot{V}_x = -T_x + F_x^{ex} \tag{8}$$

$$I_y\dot{\Omega}_y = -R_x T_x + M_y^{ex} \tag{9}$$

for the determination of the x-component of the linear velocity of the center of mass of the body, and the y-component of the angular velocity of the body. Regarding the determination of V_y and Ω_z, we either use

$$M\dot{V}_y = -T_y + F_y^{ex}, \tag{10}$$

$$I_z\dot{\Omega}_z = -M_z + M_z^{ex}, \tag{11}$$

or use the conditions that the ratios V_y/V_x and $\Omega_z \ell/V_x$ are independent of time and are known a priori. The latter conditions hold, for example, in problems involving the motion of a wheel.

We need to prescribe initial conditions for τ, V_x, Ω_y and if necessary for V_y and Ω_z that are consistent with the inequality (1).

3 Mathematical Analysis of the Problem

3a Asymptotic Estimates of Variables. We introduce nondimensional variables as follows:

$$\hat{x} = x/\ell,\ \hat{y} = y/\ell,\ \hat{\mathbf{u}}^b = \mathbf{u}^b\,\mathrm{E}_c/(\ell p_c),\ \hat{\mathbf{u}}^s = \mathbf{u}^s\mathrm{E}_c/(\ell p_c),$$
$$\hat{\mathbf{s}} = \mathbf{s}\,\mathrm{E}_c\,T_c/(p_c R_x),\ \hat{\mathbf{v}} = \mathbf{v}\,\mathrm{E}_c T_c/(p_c R_x), \tag{12}$$
$$\hat{V}_x = V_x T_c/R_x,\ \hat{V}_y = V_y\,\mathrm{E}_c T_c/p_c R_x,\ \hat{\Omega}_y = \Omega_y\,T_c,$$
$$\hat{\Omega}_z = \Omega_z\,\mathrm{E}_c T_c \ell/(p_c R_x),\ \hat{I}_x = I_x/MR_x^2,\ \hat{I}_y = I_y/MR_x^2,$$
$$\hat{\mathbf{F}}^{ex} = \mathbf{F}^{ex}/(p_c\ell^2),\ \hat{\mathbf{M}}^{ex} = \mathbf{M}^{ex}/(p_c\ell^2 R_x).$$

The characteristic time, T_c, of the process is determined by

$$T_c = (MR_x/p_c\ell^2)^{1/2}, \tag{13}$$

and the characteristic rigidity, E_c, of the system is defined by

$$4/\mathrm{E}_c = 1/\mathrm{E}^b + 1/\mathrm{E}^s + 1/k_b + 1/k_s. \tag{14}$$

Nomenclature

- A = contact area
- E^b, E^s = Young's moduli for the material of the body and the substrate
- G^b, G^s = shear moduli for the material of the body and the substrate
- I_y, I_z = components of the moment of inertia of the body about y- and z-axes
- M = mass of the body
- k_b, k_s, k_1, k_2 = flexibilities, of thin layers covering the body and the substrate, at a point of the contact surface
- P = resultant external normal force
- $\mathbf{F}^{ex}$, $\mathbf{M}^{ex}$ = externally applied forces and moments on the body
- R_x, R_y = radii of curvature of the body surface at the point of initial contact
- T_x, T_y = components of the resultant frictional force
- ℓ = characteristic dimension of the contact area
- $\mathbf{v}(v_x, v_y)$ = the slip velocity when the body is regarded as rigid
- $\mathbf{s}(s_x, s_y)$ = the slip velocity
- $\mathbf{V}(V_x, V_y)$ = the velocity of the center of mass of the body
- $\mathbf{u}^b$, $\mathbf{u}^s$ = tangential displacement of the body and the substrate
- $\Omega(\Omega_y, \Omega_z)$ = the angular velocity of the elastic body regarded as absolutely rigid
- t = time
- w^b, w^s = normal component of the displacement of the body and the substrate
- ν^b, ν^s = Poisson's ratios for the materials of the body and the substrate
- $\tau(\tau_x, \tau_y)$ = local frictional forces
- μ = the coefficient of friction between the two contacting bodies
- p = the pressure at a point on the contact surface
- p_c = characteristic contact pressure

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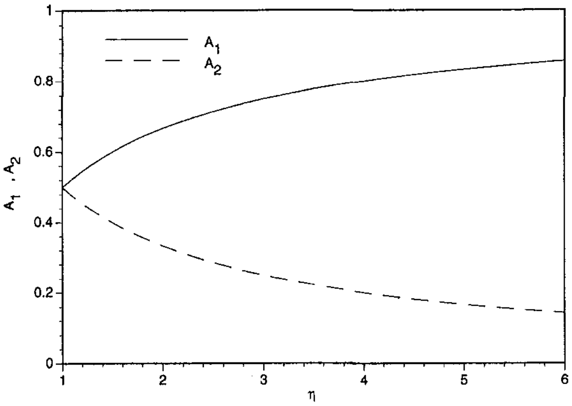

Fig. 2 Plot of functions $A_1(\eta)$ and $A_2(\eta)$

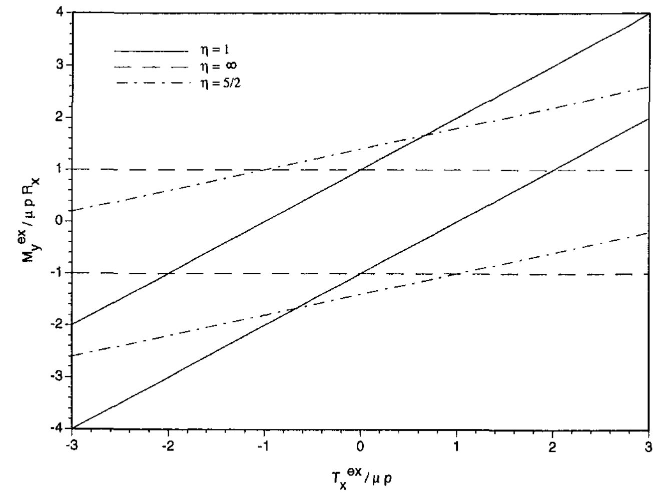

Fig. 3 The boundaries of acceptable values of T_x^{ex} and M_y^{ex} for different η

The main reason for nondimensionalizing V_x and V_y differently is that their values are of different orders of magnitude. However the values of nondimensional V_x and V_y will be of the same order of magnitude. In terms of nondimensional variables, Eq. (2) becomes

$$\mathbf{s}=\mathbf{v}-V_x\frac{\partial}{\partial x}(\mathbf{u}^b-\mathbf{u}^s)+\epsilon\frac{\partial}{\partial t}(\mathbf{u}^b-\mathbf{u}^s) \tag{15}$$

where

$$\epsilon=\ell/R_x\ll 1 \tag{16}$$

because of the local character of the contact area. In Eq. (15) and henceforth we have dropped the superimposed hat to indicate nondimensional variables. From the contact geometry, we have

$$\frac{\ell^2}{R_x}\sim|\mathbf{u}^s|,\ |\mathbf{u}^b|\sim\frac{p_c}{\mathrm{E}_c}\ell, \tag{17}$$

and hence

$$\frac{p_c}{\mathrm{E}_c}\sim\frac{\ell}{R_x}\sim\epsilon. \tag{18}$$

3b Analysis of the Transient Phase. In order that each term on the right-hand side of Eq. (15) can be of the same order of magnitude, we must have

$$t\sim\epsilon. \tag{19}$$

Therefore, we have the boundary layer effect in time, and the transient process occurs for the initial short period. During this brief time interval, the frictional forces and slip velocities are quickly redistributed. The evolution of frictional forces is determined by Eqs. (5) and (6) wherein the slip velocity is determined by (15). Equation (8) implies that in time $t\sim\epsilon$, the change in the velocity of the center of mass is also of order ϵ. Hence, neglecting small terms of higher order, we can replace V_x in (15) by its value at time $t=0$.

The equation giving the evolution of the rigid body slip velocity components and obtained from Eqs. (8) and (9) is

$$\frac{p_c}{\mathrm{E}_c}\dot{v}_x^o=-(1+\eta)\,T_x+T_x^{ex}-\eta M_y^{ex}, \tag{20}$$

$$\eta=\frac{MR_x^2}{I_y}. \tag{21}$$

Recalling estimates (18) and keeping the leading terms in (20) we obtain the following equation for the evolution of v_x^o in the initial transient phase

$$T_x\left(\frac{v_x^o}{V_x(o)},\frac{V_y}{V_x(o)},\frac{\Omega_z}{V_x(o)}\right)=-A_1M_y^{ex}(o)+A_2T_x^{ex}(o), \tag{22}$$

where

$$A_1=\eta/(1+\eta),\ A_2=1/(1+\eta). \tag{23}$$

In Eq. (22), $T_x(.,\ .,\ .)$ is a function of the indicated arguments. If the components V_y and Ω_z are not independent of time, we determine them from the following two equations

$$T_y\left(\frac{v_x^o}{V_x(o)},\frac{V_y}{V_x(o)},\frac{\Omega_z}{V_x(o)}\right)=F_y^{ex}(o), \tag{24}$$

$$M_z\left(\frac{v_x^o}{V_x(o)},\frac{V_y}{V_x(o)},\frac{\Omega_z}{V_x(o)}\right)=M_z^{ex}(o), \tag{25}$$

which can be derived from Eqs. (10) and (11).

Functions $A_1(\eta)$ and $A_2(\eta)$ for $1\le\eta\le\infty$ are plotted in Fig. 2. The values $\eta=\infty$ and $\eta=1$ correspond, respectively, to the cases when body's entire mass is concentrated on its centroidal axis or is distributed on a ring of radius R_x.

3c Analysis of Quasistationary Phase. In this phase we neglect the last term involving the partial derivative with respect to time t on the right-hand side of Eq. (15). Hence the contact problem becomes quasistationary. However, the problem is substantially evolutionary because of the effect of inertia forces. Combining the dynamics equations, we obtain

$$T_x\left(\frac{v_x^o}{V_x(t)},\frac{V_y}{V_x(t)},\frac{\Omega_z}{V_x(t)}\right)=-A_1M_y^{ex}(t)+A_2T_x^{ex}(t) \tag{26}$$

This restricts the magnitudes of M_y^{ex} and F_x^{ex} because of the inequality

$$|T_x|\le\mu P. \tag{27}$$

Figure 3 gives boundaries of acceptable areas in the space (T_x^{ex}, M_y^{ex}) for the cases $\eta=1$, ∞ and $5/2$; $\eta=5/2$ corresponds to a homogeneous ellipsoid.

Regarding (26) as the equation for v_x^o and integrating Eq. (8) for the x-component of the velocity of the center of mass, we obtain

$$v_x^o(t)=[V(o)+A_1\int_0^t(M_y^{ex}+T_x^{ex})dt]\cdot T_x^{-1}\,(-A_1M_y^{ex}+A_2T_x^{ex}) \tag{28}$$

for the longitudinal component of the rigid slip velocity. In Eq. (28) T_x^{-1} is the inverse of the function $T_x(v_x^o/V_x)$, which is determined by the solution of the contact problem for the frictional forces. For the example problem discussed in Section 4, we give an explicit expression for the function $T_x(v_x^o/V_x)$.

If V_y/V_x and Ω_z/V_x are not independent of time t, then we use

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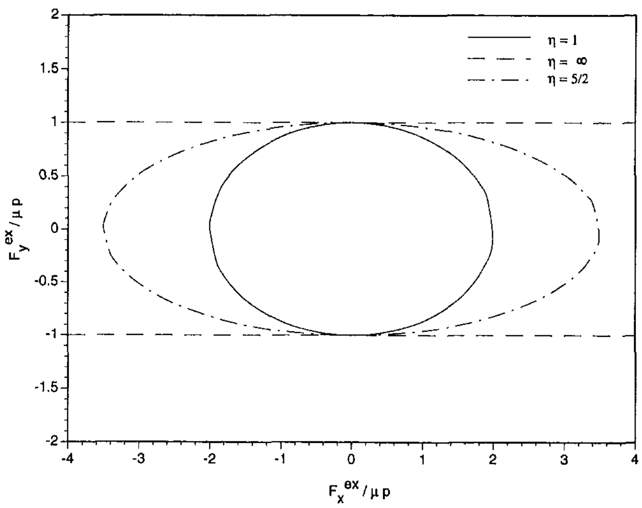

Fig. 4 The boundaries of acceptable values of F_x^{ex} and F_y^{ex} for different η

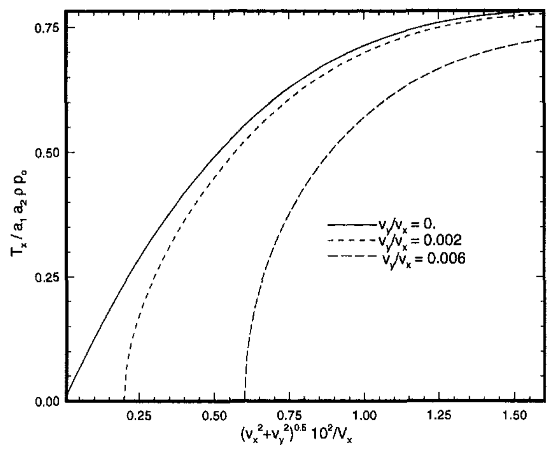

Fig. 5 The longitudinal component of the resultant frictional force as a function of $(v_x^2+v_y^2)^{1/2}/V_x$ for different values of v_y/V_x

$$F_y = F_y^{ex}(t), \tag{29}$$

$$M_z = M_z^{ex}(t) \tag{30}$$

to evaluate V_y and Ω_z. Equations (29) and (30) also restrict the magnitudes of external forces and moments. In Fig. 4, we depict boundaries of acceptable areas in the space (F_x^{ex}, F_y^{ex}) for the case $M_z^{ex}=0$ and $\eta=1$, ∞ and $5/2$. We note that Eqs. (22), (24), and (25) transform to (26), (29), and (30), respectively, in the intermediate zone belonging to both the transient and quasistationary phases. Spector and Batra (1994) have discussed conditions needed to match the solutions from the initial transient phase and the subsequent quasistationary phase.

4 An Example

In order to illustrate some of the foregoing ideas, we study the rolling/sliding of an ellipsoid of revolution under the action of a normal force P, tangential force F_x^{ex} and the moment M_y^{ex}. We assume that the body does not spin, the ratio V_y/V_x is known and fixed in time, and the rigidity of the surface layer covering the body and the substrate is much less than that of either the body or the substrate. For this case, Eqs. (6) and (7) simplify to

$$\mathbf{u}^b = -k_b\boldsymbol{\tau}, \quad \mathbf{u}^s = k_s\boldsymbol{\tau}. \tag{31}$$

Suppose that

$$w^b = k_1 p, \quad w^s = -k_2 p. \tag{32}$$

Equations $(32)_1$ and $(32)_2$ imply that the pressure distribution, p, is given by

$$p = p_o\left(1-\frac{x^2}{a_1^2}-\frac{y^2}{a_2^2}\right), \tag{33}$$

where

$$a_1=(2R_x\delta)^{1/2}, \quad a_2=(2R_y\delta)^{1/2},$$
$$p_o=\frac{\delta}{k_1+k_2}, \quad \delta=\left[\frac{P(k_1+k_2)}{\pi}\right]^{1/2}(R_xR_y)^{-1/4}. \tag{34}$$

Using equations

$$\frac{\partial\tau_x}{\partial x}=-v_x/V_x(k_b+k_s), \quad \frac{\partial\tau_y}{\partial x}=-v_y/V_x(k_b+k_s) \tag{35}$$

which hold at points in the adhesion subarea, we obtain

$$\tau_x=\begin{cases}(\ell(y)-x)\dfrac{v_x}{V_x(k_b+k_s)} & \text{if } \ell_a(y)\le x\le \ell(y),\\ \mu p_o\dfrac{v_x}{(v_x^2+v_y^2)}\left(1-\dfrac{x^2}{a_1^2}-\dfrac{y^2}{a_2^2}\right) & \text{if } -\ell(y)\le x\le \ell_a(y),\end{cases} \tag{36}$$

$$\tau_y=\frac{v_y}{v_x}\tau_x, \quad \ell(y)=a_1\left(1-\frac{y^2}{a_2^2}\right)^{1/2}. \tag{37}$$

Functions $\ell_a(y)$ and $\ell(y)$ determine, respectively, the coordinates of the left and right edges of the intersection of $y=\text{const}$ with the adhesion zones. These two functions are related as

$$\ell_a(y)=\alpha-\ell(y), \tag{38}$$

$$\alpha=\frac{2R_x(k_1+k_2)}{(k_b+k_s)\mu}\frac{(v_x^2+v_y^2)^{1/2}}{V_x}. \tag{39}$$

The integration of (36) and (37) over the contact area yields the following expressions for the resultant frictional force,

$$T_x=\mu p_o a_1 a_2\frac{v_x}{(v_x^2+v_y^2)^{1/2}}H(\beta) \tag{40}$$

$$T_y=\mu p_o a_1 a_2\frac{v_y}{(v_x^2+v_y^2)^{1/2}}H(\beta) \tag{41}$$

where

$$H(\beta)=\frac{\pi}{4}+\frac{\beta m(\beta)}{12}(13+0.5\beta^2) - 0.5(1+\beta^2)\arcsin m(\beta), \tag{42}$$

$$\beta=\alpha/a_1, \quad m(\beta)=(1-0.25\beta^2)^{1/2}. \tag{43}$$

Figure 5 depicts the variation of $T_x/\mu p_o a_1 a_2$ with $(v_x^2+v_y^2)^{1/2}/V_x$ for $v_y/V_x=0$, 0.002 and 0.006.

We first give numerical results for the body moving under the action of the force T_x^{ex} only, i.e., $T_y^{ex}=0$, $M_x^{ex}=M_y^{ex}=M_z^{ex}=0$. The effect of inertial properties of the body is demonstrated in Figs. 6 through 12. Figure 6 delineates the rear boundaries of adhesion subareas for $v_y/V_x=0.002$ and $\eta=10.9$, 2.92 and 1.7; Figs. 7 and 8 give the corresponding distributions of the tangential forces τ_x and τ_y. The analogous results for $v_y/V_x=0.006$ are shown in Figs. 9 through 11. In Fig. 12 we have plotted the evolution in time of the rigid slip velocity for the case $v_y/V_x=0.002$ and $\eta=10.9$, 2.92 and 1.7.

For the case of the motion of the body under the action of the tangential force and a time-dependent moment, Fig. 13 shows the evolution of the resultant frictional force for $T_x^{ex}=\mu p_o a_1 a_2$, $M_y^{ex}(t)=0.2\,\mu p_o R_x a_1 a_2\sin t$, and for $\eta=1.5$, 3, 10.

5 Discussion of Results

The mathematical analysis of the problem of rolling/sliding contact reveals that transient processes occur during a relatively short period after the application of external forces and mo-

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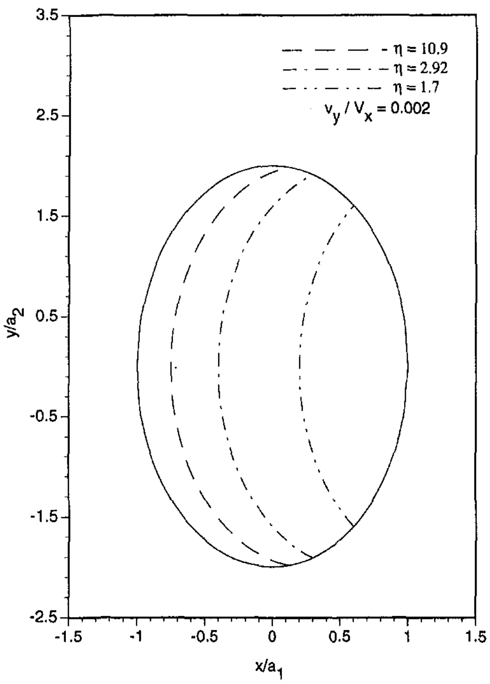

Fig. 6 The boundaries of adhesion subareas for $v_y/V_x = 0.002$ and different values of η

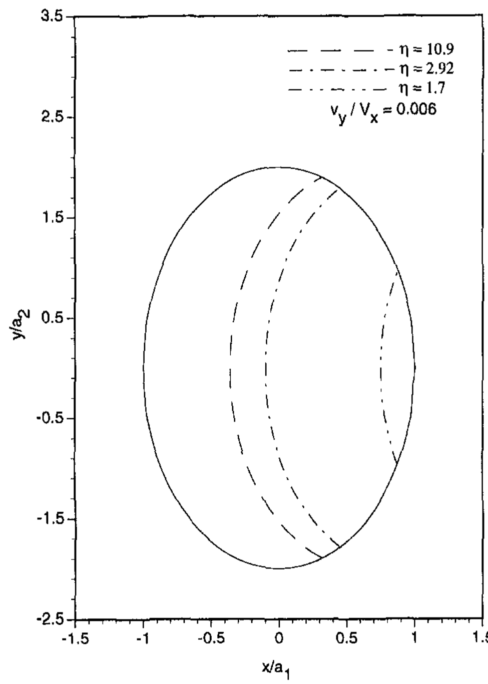

Fig. 9 The boundaries of adhesion subareas for $v_y/V_x = 0.006$ and different values of η

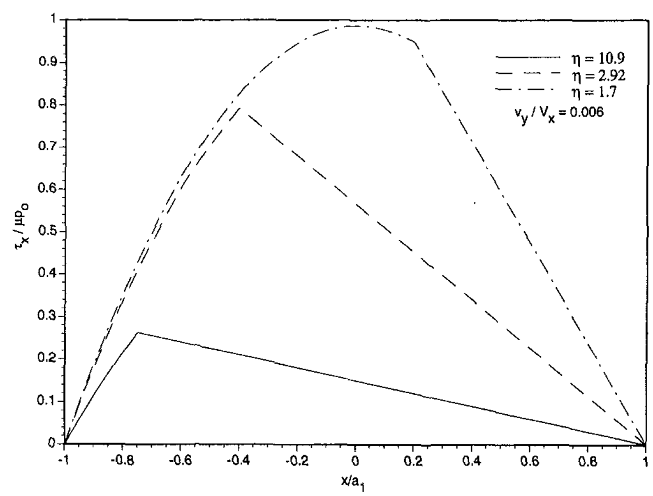

Fig. 7 The distribution of the longitudinal component of the frictional force along x-axis for $v_y/V_x = 0.002$ and different values of η

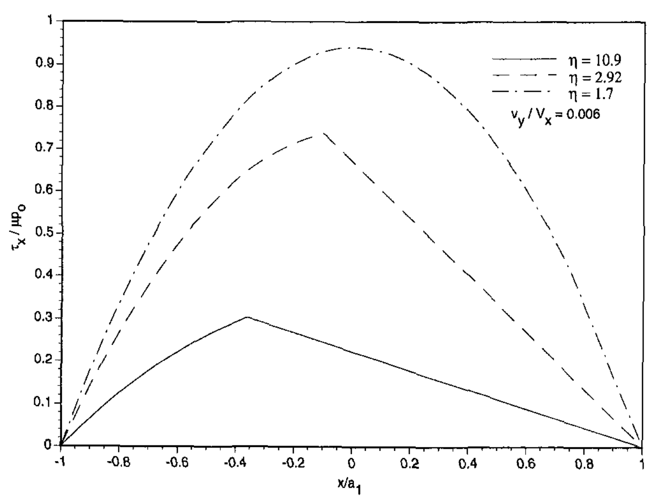

Fig. 10 The distribution of the longitudinal component of the frictional force along x-axis for $v_y/V_x = 0.006$ and different values of η

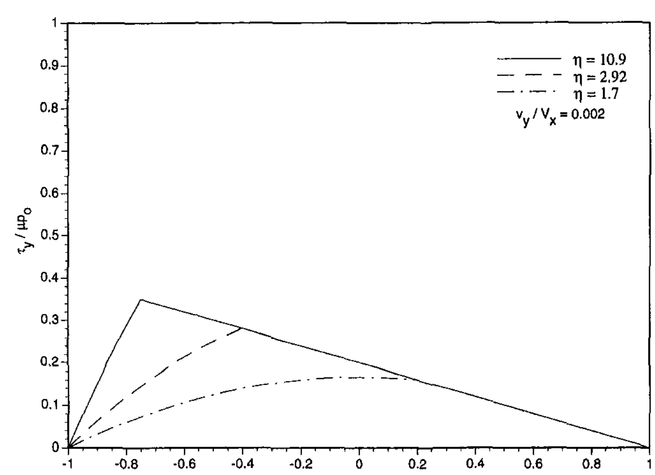

Fig. 8 The distribution of the lateral component of the frictional force along x-axis for $v_y/V_x = 0.002$ and different values of η

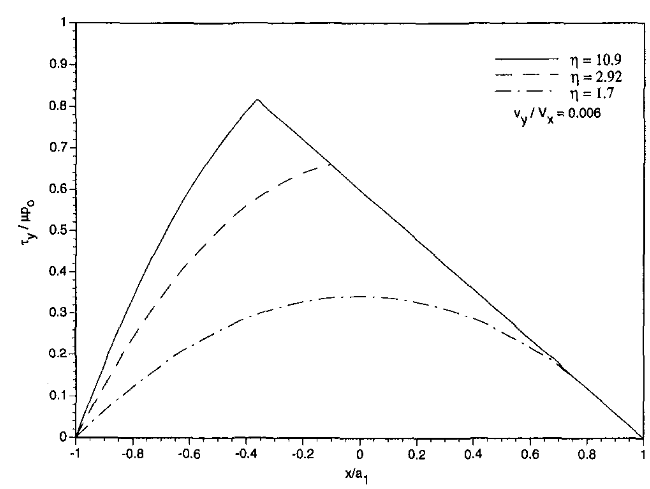

Fig. 11 The distribution of the lateral component of the frictional force along x-axis for $v_y/V_x = 0.006$ and different values of η

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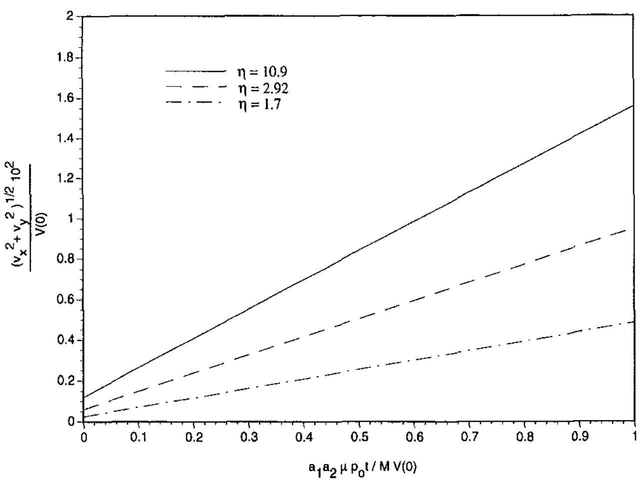

Fig. 12 The evolution in time of the rigid slip velocity for $v_y/V_x = 0.002$ and different values of η

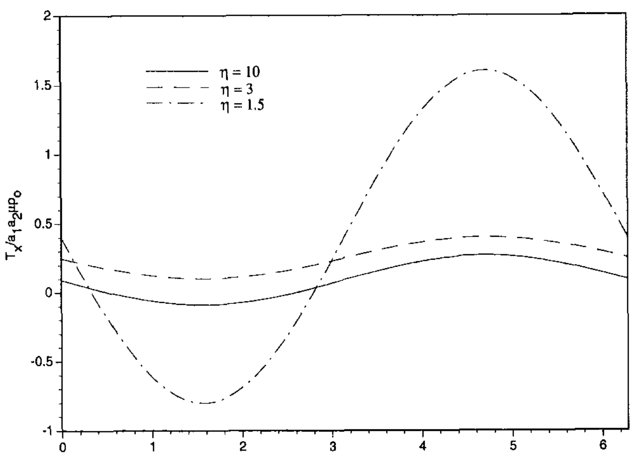

Fig. 13 Evolution of the longitudinal component of the resultant frictional force under the action of time-dependent external moment

ments. The duration of this transient phase equals the time needed for the body to traverse through a distance equal to several times the width of the contact zone. This explains Kalker's numerical results for the case of two cylinders rolling over each other. The asymptotic analysis suggests that the problem can be treated as quasistationary beyond this initial brief transient phase. During this quasistationary phase, the term involving the partial derivative with respect to time in the expression for the slip velocity can be neglected.

Because of condition (1) we have for the rolling/sliding regimes of interaction the asymptotically explicit relations between the components of the resultant frictional force and external forces and moments. Such relations are valid for both phases, and they match with each other in the intermediate zone. The expressions for the lateral component of the resultant frictional force and for the normal component of the resultant moment coincide with those for the static case. However, expressions (22) and (26) for the longitudinal component of the resultant frictional force are quite different from those in the static case. Formulas (22) and (26) involve the distribution of mass within the body through the parameter η and are related to the inertial properties of the body. Figure 2 shows how the distribution of mass within the body influences the values of coefficients A_1 and A_2. Figure 13 depicts the strong influence the inertial properties of the body have on the evolution of the resultant frictional force.

Equation (28) gives the rigid slip velocity in the quasistationary phase. In it the coefficients A_1 and A_2 depend upon the distribution of mass within the body. This equation is valid for both constant or time-dependent external forces and moments. In order to use (28), one needs to calculate the function $T_x\ (v_x/V_x)$ or $T_x\ (|\mathbf{v}|/V_x)$ if v_y/V_x is known. This can be done by solving a sequence of contact problems defined by Eqs. (4) and (5).

Figures 6 through 12 illustrate the effects of inertial properties and of the two-dimensional character of frictional forces on the solution variables and their evolution in time. The effect of the distribution of mass within the body is qualitatively the same for $v_y/V_x = 0.002$ and $v_y/V_x = 0.006$. The shift of internal mass towards the centroidal axis results in a higher value of η. This redistribution of mass together with the application of the external force at the center of mass results in a decrease of the magnitude of the typical slip velocity. Results plotted in Figs. 6 and 9 indicate that the rear boundaries of the adhesion subareas move back and corresponding subareas increase when η increases. The local distribution of frictional forces depends strongly upon the inertial properties of the body. The longitudinal component τ_x of the frictional force decreases with an increase in the value of η, however, the lateral component τ_y of the frictional force increases simultaneously because the absolute value of the local frictional force is fixed inside the slip subarea and is proportional to the normal pressure. Comparing results for two different levels of the lateral slip, we see that the general level of contact slip increases and the adhesion subareas shrink with an increase in the value of v_y/V_y. We also observe in Fig. 5 the decrease of longitudinal component of the resultant frictional force with an increase in the level of the lateral slip. Figure 12 elucidates that the evolution of the rigid slip velocity is more intensive for higher values of η as a result of the increase of the general slip level inside the contact area.

6 Conclusions

In frictional problems involving the rolling/sliding of a linear elastic body on a linear elastic substrate, the frictional forces redistribute rapidly during the initial transient phase whose duration equals an order of ϵ, where ϵ is the ratio of a typical width of the contact area to the typical dimension of the contacting bodies. Following this transient phase, the problem may be considered as quasistationary and the local rate of change of relative displacement may be neglected in the expression for the slip velocity. In the rolling/sliding regimes, we obtain explicit relations between the asymptotic values of the resultant frictional force and the externally applied forces and moments. The longitudinal component of the frictional force involves explicitly the characteristics of the inertial properties of the body. Higher values of the moment of inertia of the body increase the slip within the contact area, decrease the adhesion subareas, and enhance the intensity of slip evolution during the transient phase. The increase in the lateral component of the slip velocity decreases the longitudinal component of the resultant frictional force.

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APPENDIX

$$\mathbf{B}^b(p) = -\frac{1-2\nu^b}{4\pi G^b}\int_A \frac{1}{r}\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix} p\,dx'dy'$$

$$\mathbf{B}^b(\tau) = -\frac{1}{2\pi G^b}\int_A \frac{1}{r}\begin{Bmatrix}(1-\nu^b\sin^2\theta)\tau_{x'}+\nu^b\sin\theta\cos\theta\tau_{y'}\\ \nu^b\sin\theta\cos\theta\tau_{x'}+(1-\nu^b\cos^2\theta)\tau_{y'}\end{Bmatrix} dx'dy'$$

$$\mathbf{B}^s(p) = \frac{1-2\nu^s}{4\pi G^s}\int_A \frac{1}{r}\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix} p\,dx'dy'$$

$$\mathbf{B}^s(\tau) = \frac{1}{2\pi G^s}\int_A \frac{1}{r}\begin{Bmatrix}(1-\nu^s\sin^2\theta)\tau_{x'}+\nu^s\sin\theta\cos\theta\tau_{y'}\\ \nu^s\sin\theta\cos\theta\tau_{x'}+(1-\nu^s\cos^2\theta)\tau_{y'}\end{Bmatrix} dx'dy'$$

where

$$r=[(x-x')^2+(y-y')^2]^{1/2}$$

$$\cos\theta=\frac{x-x'}{r},\ \sin\theta=\frac{y-y'}{r}$$

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