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On the Asymptotic Stability of an Equilibrium Solution of the Boussinesq-Equations

Es wird gezeigt, daß eine Gleichgewichtslösung der Boussinesq-Gleichungen unter gemischten Randbedingungen asymptotisch stabil im Mittel ist.

It is shown that an equilibrium solution of the Boussinesq equations under mixed boundary conditions is asymptotically stable in the mean.

Показывается, что некоторое решение равновесия уравнений Буссинеска со смешанными краевыми условиями асимптотически стабильно в среднем.

1. Introduction

In [1] I studied the asymptotic stability of an equilibrium solution of coupled nonlinear equations governing the thermomechanical deformations of an incompressible Navier-Stokes-Fourier fluid. Therein I assumed that the mass density following a particle was constant so there was no buoyancy force. I now consider the case where the mass density depends upon the temperature but assume that the thermomechanical deformations of the viscous fluid are governed by the Boussinesq equations. These equations involve a coupling of the internal energy to the kinetic energy by the action of buoyance.

The problem of the stability of the Boussinesq equations has been studied extensively by JOSEPH [2, 3]. He studied the stability of a solution of these equations under the following boundary conditions: the velocity is assigned on the whole boundary and the thermal boundary condition is either of the first type or of the third type. The stability criterion obtained by JOSEPH depends both on a Rayleigh number and a Reynolds number. For an equilibrium solution both these numbers are zero. It follows from the general results proved in [2] that the equilibrium solution of the Boussinesq equations is asymptotically stable under the boundary conditions considered therein.

Here I study the asymptotic stability of an equilibrium solution of the Boussinesq equations under mixed boundary conditions and allow for the possibility that except for a material surface whose particles always have zero velocity, the boundary conditions on the remaining bounding surface may alternate between that of zero velocity and constant hydrostatic pressure. For the thermal boundary conditions I do not require that there be a material surface on which the temperature is assigned for all times.

2. Formulation of the Problem

Consider a viscous fluid at rest in a closed rigid vessel. Assume that the fluid does not fill the container and that in the reference configuration the fluid occupies a smooth and bounded region R with a smooth boundary ∂R. I assume that R is smooth enough to apply the divergence theorem, the Poincaré inequality [4, 5] and the KORN inequality [4]. Let $\boldsymbol{\chi}(R, t)$ denote a mapping of R into the present configuration so that $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$ gives in rectangular Cartesian coordinates the present position of a particle which occupied the place $\boldsymbol{X}$ in the reference configuration. The Boussinesq equations governing the thermomechanical deformations of the fluid are [6, pp. 16—18]

$$\begin{aligned} &\operatorname{div} \boldsymbol{v} = 0\,, \\ &\varrho \dot{\boldsymbol{v}} = \operatorname{div} \boldsymbol{T} + \varrho[1 \quad \alpha(\theta \quad \hat{\theta})]\, \boldsymbol{g}\,, \\ &\dot{\theta} = \varkappa \nabla^2 \theta\,, \end{aligned} \tag{2.1}$$

where

$$\boldsymbol{T} \qquad p\boldsymbol{1} + 2\mu \boldsymbol{D}$$

Here $\boldsymbol{v} \equiv \dot{\boldsymbol{x}}(\boldsymbol{X}, t) = \dfrac{\partial \boldsymbol{x}(\boldsymbol{X}, t)}{\partial t}$ denotes the velocity of $\boldsymbol{X}$ at time t, $\boldsymbol{T}$ denotes the Cauchy stress tensor and $\hat{\theta}$ is the reference temperature at which the material properties ϱ, μ, α and $\varkappa$ are evaluated. ϱ, μ, α and $\varkappa$ are, respectively, the density, the shear viscosity, the coefficient of thermal expansion and the thermometric coefficient. For the sake of simplicity all of these are assumed to be positive constants. The analysis can be modified (e.g. see [1]) to apply to the case when these depend upon $\boldsymbol{X}$. Further in (2.1) p is the arbitrary hydrostatic pressure, $\boldsymbol{D} \equiv (\operatorname{grad} \boldsymbol{v} + (\operatorname{grad} \boldsymbol{v})^T)/2$ is the strain-rate tensor and $\boldsymbol{g}$ denotes the gravity vector. The term $\alpha(\theta - \hat{\theta})\, \boldsymbol{g}$ represents the buoyancy force. Because of this term the apparent body force in $(2.1)_2$ is non-conservative.

Let the fluid be given an arbitrary initial disturbance; and let the following thermomechanical boundary conditions be maintained subsequently.

$$
\begin{aligned}
&\boldsymbol{v}(\boldsymbol{x}, t) = 0\,, && (\boldsymbol{x}, t) \in \boldsymbol{\chi}(\partial_1 R(t), t) \times (0, t)\,,\\
&\boldsymbol{T}(\boldsymbol{x}, t)\,\boldsymbol{n}(\boldsymbol{x}, t) = -p_0 \boldsymbol{n}(\boldsymbol{x}, t)\,, && (\boldsymbol{x}, t) \in \boldsymbol{\chi}(\partial_1^c R, t) \times (0, t)\,,\\
&\theta(\boldsymbol{x}, t) = \theta_0\,, && (\boldsymbol{x}, t) \in \boldsymbol{\chi}(\partial_2 R(t), t) \times (0, t)\,,\\
&-\varkappa \operatorname{grad}\theta \cdot \boldsymbol{n}(\boldsymbol{x}, t) = b(\theta - \theta_0)\,, && (\boldsymbol{x}, t) \in \boldsymbol{\chi}(\partial_2^c R, t) \times (0, t)\,.
\end{aligned}
\tag{2.2}
$$

Here $\boldsymbol{n}(\boldsymbol{x}, t)$ is an outward unit normal in the present configuration to the boundary $\boldsymbol{\chi}(\partial R, t)$ of $\boldsymbol{\chi}(R, t)$, $\partial_1 R$ and $\partial_2 R$ denote subsets of ∂R and $\partial_1^c R \equiv \partial R - \partial_1 R$. The mechanical boundary condition $(2.2)_1$ states that the fluid adheres to the walls of the container and the condition $(2.2)_2$ implies that the part of the boundary of the fluid not in contact with the walls is subjected to an uniform hydrostatic pressure. In order that heat may radiate from the fluid into the surroundings when the former is at a temperature higher than the latter, b should be positive. When the 2-dimensional measure of the set $\bigcap\limits_{t>0} \boldsymbol{\chi}(\partial_1 R(t), t)$ is positive and $\boldsymbol{v} \in L^2(\boldsymbol{\chi}(R, t))$, $\operatorname{grad} \boldsymbol{v} \in L^2(\boldsymbol{\chi}(R, t))$, one can conclude from the Poincaré inequality [4] and the Korn inequality [4] that

$$
\int\limits_{\chi(R,t)} v^2 \,\mathrm{d}V \leqq k_1(t) \int\limits_{\chi(R,t)} \operatorname{tr} \boldsymbol{D}^2 \,\mathrm{d}V\,. \tag{2.3}
$$

Also, if $\theta \in L^2(\boldsymbol{\chi}(R, t))$, $\operatorname{grad}\theta \in L^2(\boldsymbol{\chi}(R, t))$, then the use* of Poincaré's inequality [4, 5] leads to

$$
\int\limits_{\chi(R,t)} (\theta - \theta_0)^2 \,\mathrm{d}V \leqq k_2(t) \Big[\int\limits_{\chi(\partial R,t)} (\theta - \theta_0)^2 \,\mathrm{d}A + \int\limits_{\chi(R,t)} |\operatorname{grad}\theta|^2 \,\mathrm{d}V \Big]\,. \tag{2.4}
$$

In (2.3) and (2.4), $k_1(t)$ and $k_2(t)$ are positive valued functions of t whose values depend, respectively, upon $\boldsymbol{\chi}(R, t)$, $\boldsymbol{\chi}(\partial_1 R(t), t)$ and $\boldsymbol{\chi}(R, t)$, $\boldsymbol{\chi}(\partial_2 R(t), t)$.

Hereafter, instead of (2.1) and (2.2) I shall be dealing with the following integral equations (2.5) derived from them. Taking the inner product of $(2.1)_2$ with $\boldsymbol{v}$, multiplying $(2.1)_3$ by $(\theta - \theta_0)$, integrating the resulting equations over $\boldsymbol{\chi}(R, t)$, and then simplifying by using the divergence theorem, the boundary conditions (2.2) and the equation $(2.1)_1$, I obtain

$$
\begin{aligned}
&\dot{K} \quad 4\nu \int\limits_{\chi(R,t)} \operatorname{tr} \boldsymbol{D}^2 \,\mathrm{d}V + 2[1 - \alpha(\theta_0 - \hat{\theta})]\,\boldsymbol{g} \cdot \int\limits_{\chi(R,t)} \boldsymbol{v} \,\mathrm{d}V - 2\alpha \boldsymbol{g} \cdot \int\limits_{\chi(R,t)} (\theta \quad \theta_0)\,\boldsymbol{v} \,\mathrm{d}V\\
&\dot{H} \quad -2b \int\limits_{\chi(\partial_2^c R,t)} (\theta - \theta_0)^2 \,\mathrm{d}A - 2\varkappa \int\limits_{\chi(R,t)} |\operatorname{grad}(\theta - \theta_0)|^2 \,\mathrm{d}V\,,
\end{aligned}
$$

where

$$
\begin{aligned}
K(t) &\equiv \int\limits_{\chi(R,t)} v^2 \,\mathrm{d}V\\
H(t) &\equiv \int\limits_{\chi(R,t)} (\theta(t) \quad \theta_0)^2 \,\mathrm{d}V\,,
\end{aligned}
$$

and $\nu \equiv \mu/\varrho$ is the kinematic viscosity of the fluid. Let S denote the set of initial conditions for which there exists a solution $(\boldsymbol{v}, \theta)$ of the integral equations (2.5) such that for every $t > 0$,

(i) the mapping $\boldsymbol{\chi}$ of R into the present configuration is twice continuously differentiable with respect to $\boldsymbol{X}$ and t,

$$K \leqq B_2, \quad \operatorname{grad} \boldsymbol{v} \in L^2(\boldsymbol{\chi}(R, t))\,,$$

(ii) θ is continuously differentiable with respect to $\boldsymbol{x}$ and t, $\theta \in L^2(\boldsymbol{\chi}(R, t))$, $\operatorname{grad}\theta \in L^2(\boldsymbol{\chi}(R, t))$,

(iii) $1/K_i \equiv \sup\limits_{t>0} k_i(t)$, $(i = 1, 2)$, is finite, and

(iv) $(\boldsymbol{v}, \theta)$ satisfies (2.5) and suitable initial conditions.

Such a solution of (2.5) can be regarded as a weak solution of (2.1) and (2.2). The requirement $K \leqq B_2$ implies that the total kinetic energy of the fluid stays bounded. I assume that S is non-empty and state the theorem I wish to prove below.

Theorem: *For every initial disturbance which belongs to S, the solution $(\boldsymbol{v}, \theta)$ of (2.5) exhibits the behavior:*

$$
\begin{aligned}
&K(t) \to 0, \text{ as } t \to \infty\,,\\
&H(t) \to 0 \textit{ exponentially as } t \to \infty\,,
\end{aligned}
\tag{2.7}
$$

provided $\varkappa$, ν, α and the 2-dimensional measure of the set $\bigcap\limits_{t>0} \boldsymbol{\chi}(\partial_1 R(t), t)$ are positive.

*) For the case when either $\bigcap\limits_{t>0} \chi(\partial_2 R(t), t) \neq \emptyset$ or $b > 0$, see [1]. That (2.4) holds even when $\bigcap\limits_{t>0} \chi(\partial_2 R(t), t) = \emptyset$ and $b = 0$ is shown in [7].

3. Proof of the Theorem

Because of $(2.2)_3$ I can replace the region $\chi(\partial_2^c R, t)$ of integration in the first integral on the right-hand side of $(2.5)_2$ by $\chi(\partial R, t)$. Note that $(\theta - \theta_0) \in L^2(\chi(\partial R, t))$ follows from the requirement (ii) in the definition of a weak solution of (2.5) and the theorem of trace [8]. Setting $c \equiv 2\min(b, \varkappa)$, using (2.4) and $k_2(t) \leqq 1/K_2$ I obtain

$$\dot{H} \leqq -cK_2 H(t) . \tag{3.1}$$

The smoothness required of $\theta(X, t)$ stated in the definition of a solution of (2.5) implies that $\dot{H}$ is bounded. Hence H is of bounded variation on $(0, T)$ and thus $\dot{H} \in L^1(0, T)$ where T is an arbitrary real positive number. Integration of (3.1) over $(0, T)$ yields

$$H(T) \leqq H(0)\,\mathrm{e}^{-cK_2 T}$$

which implies $(2.7)_2$. Now using the Cauchy-Schwarz inequality, I bound the last integral on the right-hand side of $(2.5)_1$ as follows.

$$|\boldsymbol{g} \cdot \textstyle\int (\theta - \theta_0)\, \boldsymbol{v}\, \mathrm{d}V|^2 \leqq |\boldsymbol{g}|^2 \int (\theta - \theta_0)^2\, \mathrm{d}V \int v^2\, \mathrm{d}V, \leqq |\boldsymbol{g}|^2 B_2 H(0)\, \mathrm{e}^{-cK_2 t} . \tag{3.2}$$

Using the arguments needed to conclude that $\dot{H} \in L^1(0, T)$ I obtain $\dot{K} \in L^1(0, t)$. An integration of $(2.5)_1$ over $(0, T)$ and the use of (3.2) give

$$\begin{aligned} K(T) \leqq K(0) - 4\nu \int_0^T \mathrm{d}t \int \operatorname{tr} \boldsymbol{D}^2\, \mathrm{d}V + 2[1 - \alpha(\theta_0 - \hat{\theta})]\, \boldsymbol{g} \cdot \int \boldsymbol{x}\, \mathrm{d}V\Big|_0^T + \\ + \frac{4\alpha}{cK_2} |\boldsymbol{g}|\, (B_2 H(0))^{1/2}\, (1 - \mathrm{e}^{-cK_2 T/2}) . \end{aligned} \tag{3.3}$$

Since the vessel is closed $|\int \boldsymbol{x}\, \mathrm{d}V| \leqq$ constant. I conclude from (3.3) that

$$\int \operatorname{tr} \boldsymbol{D}^2\, \mathrm{d}V \in L^1(0, \infty) \tag{3.4}$$

for otherwise $K(T)$ would be negative for some large value of T. (3.4), (2.3) and $k_1(t) \leqq 1/K_1$ imply that

$$K(t) \in L^1(0, \infty) .$$

Now from (3.4), (3.2), (2.7) and $(2.5)_1$ it follows that

$$\dot{K}(t) \in L^1(0, \infty) .$$

Thus $K(t)$ is uniformly continuous in t and this when combined with (3.5) gives $(2.7)_1$.

4. Remarks

The result $(2.7)_1$ is weaker than the one obtainable from the more general results of JOSEPH [2]. A reason for this is that whereas for the mechanical boundary condition considered by JOSEPH the potential energy of the fluid remains constant for all times $t > 0$; this is not so for the mixed boundary conditions considered here since there is the possibility of the exchange of the mechanical energy between the kinetic and the potential parts.

It is clear from the preceeding analysis that a weak solution of $(2.1)_3$ under the boundary conditions $(2.2)_{3,4}$ exhibits the behavior depicted by $(2.7)_2$. In particular, $(2.7)_2$ holds for a rigid heat conductor. That a similar result holds for a nonlinear heat conductor is shown in [9].

Acknowledgements

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