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Asymptotic Stability of an Equilibrium Solution of a Hyperbolic Heat Equation

Introduction

In recent years there has been considerable interest [1–6] in proposing a theory of heat conduction which allows for the propagation of weak thermal disturbances i.e. discontinuities in second order derivatives of temperature with finite speed. Of the various theories proposed, some [4–6] can be derived from the equation expressing the balance of internal energy using modern continuum thermodynamics arguments. In [4] Bogy and Naghdi study rate dependent heat conductors and show that, under certain conditions, weak thermal disturbances can travel with finite speed. However, in the theory linearized about a uniform temperature distribution thermal waves either do not propagate or else they propagate with infinite speed. If these rate-dependent materials are studied according to an entropy inequality proposed by Müller [5] or that by Green and Laws [6] then one obtains in the theory linearized about a uniform temperature distribution the equation

$$b\ddot{\sigma} + c\dot{\sigma} = \operatorname{div}(K \operatorname{grad} \sigma) .$$

In this equation σ denotes the temperature, a superimposed dot indicates material time differentiation, c is the specific heat, K is the thermal conductivity tensor and b is a material constant. We assume that the body is homogeneous and therefore take b, c and K to be constants throughout the body. The entropy inequality implies that

$$c \geqq 0 , \qquad \xi \cdot \boldsymbol{K}\xi \geqq 0 \; \forall \xi ,$$

and allows for the possibility that b be positive. When $c > 0$, $b > 0$ and $\boldsymbol{K}$ is positive definite, the above equation is a hyperbolic equation and therefore allows for the propagation of thermal waves with finite speed.

For the heat conductor governed by the classical parabolic linear equation [7, 8] and a nonlinear parabolic equation [9], Batra proved that the temperature field in the body approaches in L^2-norm the uniform temperature field under rather general boundary conditions. The motivation for studying the problem in [7] and for the present problem is the desire to show that for dissipative loading devices*) constitutive quantities do not depend upon the initial state of the loading device provided that the entire past history of its boundary conditions is known. For this purpose, as explained in [7], it is sufficient to prove the following

Theorem: *The equation*

$$b\ddot{\sigma}(x, t) + c\dot{\sigma}(x, t) = (K_{ij}\sigma_{,j}(x, t))_{,i} \qquad (x, t) \in R \times (0, t) , \tag{1}$$

in which

$$b > 0 , \qquad c > 0 , \qquad K_{ij}\xi_i\xi_j > 0 , \qquad \forall \xi \neq \boldsymbol{0} , \tag{2}$$

under the boundary conditions

$$\begin{aligned} &\sigma(x, t) = 0 , \qquad (x, t) \in \partial_1 R \times (0, t) , \\ &\frac{\partial\sigma}{\partial\nu}(x, t) = 0 , \qquad (x, t) \in \partial_2 R \times (0, t) , \end{aligned} \tag{3}$$

and the initial conditions

$$\begin{aligned} &\sigma(x, 0) = \sigma_0(x) , \qquad x \in R , \\ &\dot{\sigma}(x, 0) = \dot{\sigma}_0(x) , \qquad x \in R , \end{aligned} \tag{4}$$

has a weak solution which exhibits the behavior

$$\int \sigma^2(x, t) \, \mathrm{d}V \leqq O(\mathrm{e}^{-\gamma t}) , \qquad \gamma > 0 \quad \text{as} \quad t \to \infty , \tag{5}$$

provided that $\sigma_0 \in L^2(R)$, $\dot{\sigma}_0 \in L^2(R)$ *and when* $\partial_1 R = \Phi$, *the initial conditions also satisfy*

$$\int \sigma_0(x) \, \mathrm{d}V = \int \dot{\sigma}_0(x) \, \mathrm{d}V = 0 . \tag{6}$$

Here $R \subset E^m$ is a smooth m-dimensional manifold with a smooth boundary ∂R, $\partial_1 R$ and $\partial_2 R$ are disjoint parts of ∂R which together make ∂R, x denotes the position of a point with respect to rectangular Cartesian coordinates, ν is an outward directed unit vector normal to the boundary at $x \in \partial R$, $\mathrm{d}V$ is a volume measure on E^m, and a comma followed by an index j implies differentiation with respect to x_j.

Since we are requiring that σ_0 and $\dot{\sigma}_0$ belong only to $L^2(R)$ we can only hope to find a weak solution of (1). (E.g. see Lions [11, p. 265]). If σ_0 and $\dot{\sigma}_0$ have, respectively, two and one generalized derivatives in $L^2(R)$ then σ will have two generalized derivatives with respect to x and t in $L^2(R)$. Reviewing the physics of the problem we note that if σ_0 and $\dot{\sigma}_0$ do not have compact support contained in R, then thermal disturbances will start propagating from the boundary into the interior of the body or vice versa. Thus one cannot expect equation (1) to have a smooth solution for all times and for all initial data which belong to $L^2(R)$. In this context see also Courant and Hilbert [12, p. 673].

Proof of the Theorem

We propose to construct a solution of (1) by the Galerkin method and will need the following result (Garbedian [13]). The normalized eigenfunctions of the eigenvalue problem

$$\begin{aligned} &(K_{ij}\varphi_{,j}(x))_{,i} = -\lambda\varphi(x) , \qquad x \in R , \\ &\varphi(x) = 0 , \qquad x \in \partial_1 R , \\ &\frac{\partial\varphi}{\partial\nu}(x) = 0 , \qquad x \in \partial_2 R , \end{aligned} \tag{7}$$

form a complete set in $L^2(R)$. Arranging the eigenvalues as $\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \ldots$ and denoting the corresponding eigenfunctions by $\varphi_1, \varphi_2, \ldots$ we have $\lambda_1 > 0$ for $\partial_1 R \neq \Phi$ and when $\partial_1 R = \Phi$, $\lambda_1 = 0$, $\varphi_1 = 1/\sqrt{V(R)}$, $\lambda_2 > 0$. Here $V(R)$ stands for the volume of R. Furthermore, the sequence λ_n has no cluster point and $\lambda_n = O(n)$ for n large and greater than one. Let

$$\sigma^N(x, t) = \sum_{n=1}^{N} h_n(t) \, \varphi_n(x) , \tag{8}$$

where φ_n is an eigenfunction of the appropriate boundary value problem. Requiring that σ^N satisfy (1) and (4), we substitute for σ^N from (8) into these equations, multiply the resulting equations by φ_n, integrate these over R and thus obtain

$$\begin{aligned} &b\ddot{h}_n + c\dot{h}_n + \lambda_n h_n = 0 , \qquad n = 1, 2, \ldots, N , \\ &h_n(0) = \int \sigma_0(x) \, \varphi_n(x) \, \mathrm{d}V \equiv f_n , \\ &\dot{h}_n(0) = \int \dot{\sigma}_0(x) \, \varphi_n(x) \, \mathrm{d}V \equiv g_n . \end{aligned} \tag{9}$$

The solution of (9) is

$$h_n(t) = \mathrm{e}^{-\frac{c}{2b}t} \left[f_n \cos \mu_n t \qquad \frac{g_n - \frac{c}{2b} f_n}{\mu_n} \sin \mu_n t \right.$$

*) See Batra [10] for the definition of a loading device and the details of derivation of its linear constitutive relations.

in which $\mu_n = \frac{1}{2b}\sqrt{4b\lambda_n - c^2}$ and it has been assumed that all λ_n are distinct.

Should two or more eigenvalues λ_n be equal, the above solution is to be suitably modified. By following arguments essentially parallel to those used by LIONS [11, p. 265] who considered the problem for which $c = 0$, we can show that $\sigma^N(x, t) \to \sigma(x, t)$ as $N \to \infty$ in the space

$$V = \{u(x, t) \mid u(x, t) \colon R \times (0, t) \to E^1, \\ \int u^2 \, dv + \int |\text{grad } u|^2 \, dV < \infty\} .$$

Therefore,

$$\sigma = \sum_{n=1}^{\infty} e^{-\frac{c}{2b}t} \left[f_n \cos \mu_n t + \frac{g_n - \frac{c}{2b} f_n}{\mu_n} \sin \mu_n t \right] \tag{10}$$

is a weak solution of (1). We note that the definitions $(9)_{2,3}$ of f_n and g_n imply that these are FOURIER coefficients of the expansion of σ_0 and $\dot{\sigma}_0$ in terms of a complete set of orthonormal functions φ_n. Hence

$$\sum_{n=1}^{\infty} f_n^2 \leq \int \sigma_0^2(x) \, dV , \qquad \sum_{n=1}^{\infty} g_n^2 \leq \int \dot{\sigma}_0^2(x) \, dV . \tag{11}$$

We now prove that σ given by (10) exhibits the behavior (5). For the case when $\lambda_1 > \frac{c^2}{4b}$, we obtain from (10), (11) and the inequality

$$(a_1 + a_2)^2 \leq 2(a_1^2 + a_2^2) , \tag{12}$$

the following

$$\int \sigma^2(x, t) \, dV \leq 2 \, e^{-\frac{c}{b}t} \left[\left(\int \sigma_0^2 \, dV \right) \left(1 + \frac{c^2}{2b^2\mu_1^2} \right) + \frac{2}{\mu_1^2} \int \dot{\sigma}_0^2 \, dV \right],$$

which is the desired result. Now let $\lambda_1 = \frac{c^2}{4b}$ and $\lambda_2 > \frac{c^2}{4b}$. Then (10) becomes

$$\sigma(x, t) = e^{-\frac{c}{2b}t} \left[\left(f_1 + \left(g_1 - \frac{c}{2b} f_1 \right) t \right) \varphi_1(x) \right] + \\ + \sum_{n=2}^{\infty} e^{-\frac{c}{2b}t} \left[\left(f_n \cos \mu_n t + \frac{g_n - \frac{c}{2b} f_n}{\mu_n} \sin \mu_n t \right) \varphi_n(x) \right],$$

and therefore

$$\int \sigma^2 \, dV \leq 2 e^{-\frac{c}{b}t} \left[\left(\int \sigma_0^2 dV \right) \left(1 + \left(\frac{1}{\mu_2^2} \quad t^2 \right) \frac{c^2}{2b^2} \right) + \\ + 2 \left(\int \dot{\sigma}_0^2 \, dV \right) \left(t^2 + \frac{1}{\mu_2} \right) \right].$$

Since $e^{-\frac{c}{b}t} t^2$ is $O\left(e^{-\frac{c}{b}t}\right)$ as $t \to \infty$, we again get the desired result (5). Now envisage that

$$\lambda_i < \frac{c^2}{4b}, \qquad i = 1, 2, \ldots , P .$$

For this case (10) becomes

$$\sigma(x, t) = \sum_{n=1}^{P} e^{-\frac{c}{2b}} \left(f_n \quad \frac{g_n \quad \overline{2b} f_n}{\bar{\mu}_n} \right) \frac{e^{\bar{\mu}_n t}}{2} \\ \left(f_n - \frac{g_n - \frac{c}{2b} f_n}{\bar{\mu}_n} \right) \frac{e^{-\bar{\mu}_n t}}{2} \Bigg] \\ + \sum_{n=P+1}^{\infty} e^{-\frac{c}{2b}t} \left[f_n \cos \mu_n t + \frac{g_n - \frac{c}{2b} f_n}{\mu_n} \sin \mu_n t \right] \varphi_n(x) , \tag{13}$$

in which $\bar{\mu}_n = \frac{1}{2b}\sqrt{c^2 - 4b\lambda_n}$. The second term can be shown to satisfy (5) by arguing the same way as was done for the first case considered above. As for the first term we obtain by using (12) repeatidly,

$$\int \left(\sum_{n=1}^{P} h_n(t) \, \varphi_n(x) \right)^2 dV \leq 2^{P-1} \, e^{-\left(\frac{c}{b} - 2\bar{\mu}_1\right)t} \\ \times \left[\left(1 + \frac{c^2}{4b^2\bar{\mu}_1^2} \right) \int \sigma_0^2 \, dV + \frac{1}{\bar{\mu}_1^2} \int \dot{\sigma}_0^2 \, dV \right]$$

Since $\lambda_1 > 0$, therefore $c/b - 2\bar{\mu}_1 > 0$ and thus the first term in (13) also decays exponentially as $t \to \infty$.

When $\partial_1 R = \Phi$, $\lambda_1 = 0$ and because of (6), $f_1 = g_1 = 0$ and the solution of (9) for $n = 1$ is $h_1 = 0$. In this case σ is given by (10) and the first term in the summation is identically zero. The preceeding arguments show that σ satisfies (5) in this case too.

References

1 CHESTER, M., Second Sound in Solids, Phys. Rev. **131**, 2013–2015, 1963.
2 ULBRICH, C. W., Exact Electric Analogy to the VERNOTTE Hypothesis, Phys. Rev. **123**, 2001–2002, 1961.
3 GURTIN, M. E. and A. C. PIPKIN, A General Theory of Heat Conduction with Finite Wave Speeds., **31**, 113–126, 1968.
4 BOGY, D. B. and P. M. NAGHDI, On Heat Conduction and Wave Propagation in Rigid Solids, J. Math. Phys., **11**, 917–923, 1970.
5 MÜLLER, I., The Coldness, a Universal Function in Thermoelastic Solids, Arch. Rat. Mech. Anal., **41**, 319–332, 1971.
6 GREEN, A. E. and N. LAWS, On the Entropy Production Inequality, Arch. Rat. Mech. Anal., **45**, 47–53, 1972.
7 BATRA, R. C., On the Fading Memory of Initial Conditions, Quart. Appl. Math., **31**, 363–371, 1973.
8 BATRA, R. C., On the Asymptotic Stability of an Equilibrium Solution of the BOUSSINESQ Equations, ZAMM, **55**, 727–729, 1975.
9 BATRA, R. C., Addendum to "A Theorem in the Theory of Incompressible NAVIER-STOKES-FOURIER fluids", Istituto Lombardo (Rend. Sc.) A, **108**, 699–704, 1974.
10 BATRA, R. C., On Non-Classical Boundary Conditions, Arch. Rat. Mech. Anal., **48**, 163–191, 1972.
11 LIONS, J. L. and E. MAGENES, Non-Homogeneous Boundary Value Problems and Applications, vol. 1, Springer-Verlag: Berlin/Heidelberg/New York 1972.
12 COURANT, R. and D. HILBERT, Methods of Mathematical Physics, Vol. II, Interscience Publishers, New York/London/Sydney, 1966.
13 GARABEDIAN, P. R., Partial Differential Equations, John Wiley & Sons, Inc., New York/London/Sydney, 1964.

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